What do we know for sure about the cuprate superconductors?

1. Macroscopic EM properties

[Ref.: Tinkham (1996 edition), Ch. 9]

In this and the next lecture, I shall address the question: How much, if anything, can we infer about the general nature of the normal and/or superconducting states of the cuprates without recourse to a specific microscopic model?[1]

Probably the single most important piece of experimental information we have on cuprate superconductivity concerns flux quantization and the Josephson effect. These experiments were done on YBCO at an early stage, and gave the results that are standard for classic superconductors, i.e. the unit of flux quantization is $h/2e$ (not e.g. h/e or $h/4e$) and the Josephson frequency-voltage relation is $\omega = 2eV/\hbar$. However, there is one subtle point that is often overlooked: the circuits used in the experiments were without exception such that the "paths" with respect to which the flux is quantized (etc.) lie entirely in the ab-plane. It is theoretically conceivable (though to my mind improbable, in view ofthe considerations below) that a direct experiment using an "all c-axis" circuit, should it be possible, would give a different result.

The significance of these results is that, according to the argument of Part I, lecture 14, they provide very strong evidence that the superconducting state of the cuprates possesses long-range order in the two-particle correlation function (and does not have it in the one-particle one), which is, crudely speaking, equivalent to the statement that the "topology" of the wave function corresponds to formation of Cooper pairs just as in the classic superconductors. If we assume, as it almost universally is done, that this result holds for the c-axis as well as for the ab-plane, then this knowledge is sufficient for us to set up a Ginzburg-Landau description in terms of an order parameter which, just as in the classic superconductors, will have the physical significance of the center-of-mass wave function of the Cooper pairs. However, in distinction to the case of a classic (isotropic) superconductor the parameters of the theory will evidently distinguish between ab-plane and c-axis.

At this point, anticipating the conclusions to be obtained in the next lecture, we might ask whether the fact that the internal state of the Cooper pairs will turn out, almost certainly, to be "exotic", that is to have a symmetry lower than that of the lattice, will effect the validity of the GL description? The answer is no, at least so long as it corresponds to a single nondegenerate irreducible representation of the crystal symmetry group (see next lecture), but the reason is quite subtle: Although the OP does in a sense possess an "orientation," that orientation is *not free to adjust itself arbitrarily*, but is pinned to the original crystal lattice, and therefore does not constitute a real "degree of freedom" which needs to be explicitly taken into account. Were the orientation free to

[1] For the purpose of this discussion, I will generally assume, unless there is a good reason not to, that experimental results which may have been obtained on a single cuprate are representative of the properties of the cuprates as a whole.

adjust, as for example in the case of the l-vector in superfluid ^{3}He-A, it would have to be incorporated explicitly and the description would become more complicated.

Thus, we proceed just as in the classic case but with allowance for the anisotropy: we treat the coordinate $\mathbf{r}$ for the moment as a continuous variable and define, just as there, a complex scalar order parameter $\Psi(\mathbf{r})$ and write the usual terms proportional to $|\Psi(\mathbf{r})|^2$ and $|\Psi(\mathbf{r})|^4$ in the free energy. The gradient term, however, must now be treated as a *tensor* quantity γ_{ab} with eigenvalues $\gamma_\parallel$, $\gamma_\perp$ corresponding to in-plane[2] and c-axis variation. Thus, the relevant form of the free energy density is

$$\begin{aligned} F\{\Psi(\mathbf{r})\} = -\alpha(T)|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta(T)|\Psi(\mathbf{r})|^4 + \\ + \sum_{\alpha\beta}\gamma_{\alpha\beta}(T)\ \left\{\left(\boldsymbol{\nabla}_\alpha + 2ie\mathbf{A}_\alpha(\mathbf{r})\right)\Psi^*(\mathbf{r})\ \left(\boldsymbol{\nabla}_\beta - 2ieA_\beta(\mathbf{r})\right)\Psi(\mathbf{r})\right\} \end{aligned} \tag{1}$$

where as usual in the limit $T \to T_c$ we assume the temperature + magnetic field dependence $\beta(T) \sim$ const $\equiv \beta$, $\gamma_{\alpha\beta}(T) \sim$ const $\equiv \gamma_{\alpha\beta}$, $\alpha(T) = \alpha_0(1 - T/T_c)$ ($\alpha_0 =$ const).

It is worth taking a moment to discuss the limits of validity of eqn. (1). Strictly speaking, it is valid only in the limits $T \to T_c$ and infinitely slow spatial variation. A generalization to arbitrary T can (as in the classic case) be simply achieved by replacing the first two terms in F by a more general function $F_{\rm loc}\{|\Psi(\mathbf{r})|^2, T\}$, and usually does not change things qualitatively. The question of the spatial variation, however, is more tricky. We recall that for a given eigenvalue γ of $\gamma_{\alpha\beta}$, the GL healing length $\xi(T)$ is given by $\xi(T) \equiv (\gamma(T)/\alpha(T))^{1/2} = \xi_0(1-T/T_c)^{-1/2}$ where $\xi_0 \equiv (\gamma_0/\alpha_0)^{1/2}$. Crudely speaking, $\xi(T)$ is the distance over which the order parameter has to bend appreciably either in amplitude or in phase before the bending energy exceeds the original condensation energy; thus, the maximum gradient of the OP that is physically realistic is of order $\xi^{-1}(T)$. The GL description will therefore be a generally valid description, at given T, if $\xi(T)$ exceeds by an appreciable margin any "microscopic" lengths in the problem (since correction terms, e.g. of the form $|\boldsymbol{\nabla}\Psi|^4$ may be expected to become appreciable when the bending is over such a microscopic length). We recall for orientation that in the standard BCS case the longest such microscopic length is the (nearly temperature-independent) pair radius ξ_p, which in BCS theory is of the same order as the prefactor ξ_0 in $\xi(T)$; thus, for $t = 1-T/T_c \ll 1$ the GL description is generally valid. In the case of a layered system like the cuprates, this argument goes over unchanged as regards the ab-plane behavior. However, in the case of c-axis bending it may turn out that the prefactor $\xi_0^{(c)}$ of $\xi^{(c)}(T)$ is only of the order of a few Å or even less, and thus, in particular, smaller than the characteristic microscopic scale of the lattice structure (i.e. the (effective) c-axis cell dimension, $\sim 6 - 15$Å). In this case, eqn. (1) will still be a valid description in the limit $T \to T_c$, but for T appreciably away from T_c it may need to be replaced by a more microscopic description: cf. below.

Before we do so, however, let us derive from eqn. (1) some simple consequences that should be valid, at least, in the limit $T \to T_c$. We first define the GL healing lengths

[2]We implicitly assume isotropy within the ab-plane; where this is not present (as in YBCO) the appropriate generalization is obvious.

$\xi_c(T)$, $\xi_{ab}(T)$ in the obvious way $\xi_c(T) \equiv (\gamma_\perp/\alpha(T))^{1/2}$, etc. and also the penetration depths $\lambda_c(T)$, $\lambda_{ab}(T)$; here it is essential to appreciate that $\lambda_c(T)$ means the penetration depth *which screens currents flowing along the c-axis*, for which the direction of penetration actually lies in the ab-plane. Now, by deriving an expression for the electric current from (1) in the standard way and inserting it in the Landau equation, we see that for given $\Psi(\mathbf{r})$ the eigenvalue of λ is proportional to $\gamma^{-1/2}$, while as we have seen the eigenvalue of ξ is proportional to $\gamma^{1/2}$. Thus we have the important relation, for any given T,

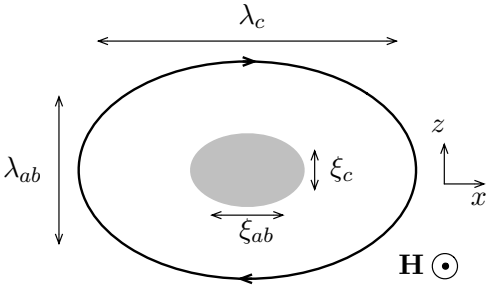
$$\xi_{ab}\lambda_{ab} = \xi_c\lambda_c = f(T) \tag{2}$$

Thus, while the healing (coherence) length along the c-axis is much smaller than that in the ab-plane, the penetration depth for currents flowing along the c-axis is much *larger*. (On the actual value of $f(T)$, see below.)

Consider now the structure of a vortex in a cuprate superconductor. If **H** is parallel to the c-axis (normal to the ab-plane), then all currents flow in the plane and the anisotropy has no effect; the theory is identical to that for an isotropic superconductor, provided that for ξ and λ we use the ab-plane values ξ_{ab} and λ_{ab} respectively. In particular, the critical field in the z-direction ($H_{c2}^{\perp}$) is given by

$$H_{c2}^{\perp}(T) = \Phi_0/2\pi\xi_{ab}^2(T) \tag{3}$$

For a field lying in the ab-plane, we can obtain the form of the vortex by scaling the coordinate axes appropriately (cf. Part I, problem 4.1(d)). Both the core and the overall size of the vortex are strongly elongated in the direction in the ab-plane perpendicular to the field; because of the relation (2) and the meaning of λ_c, we see that the *shape* of the vortex is not a function of temperature (provided we stay in the region where (1) remains valid). If we introduce, as is conventional, the anisotropy ratio[3] $\eta \equiv \xi_{ab}/\xi_c$, then this gives the ellipticity of the vortex. The upper critical field in the ab-plane ($H_{c2}^{\parallel}$) is given by the expression



$$H_{c2}^{\parallel}(T) = \Phi_0/2\pi\xi_{ab}(T)\xi_c(T) \tag{4}$$

so that

$$H_{c2}^{\parallel}(T)/H_{c2}^{\perp}(T) = \eta \tag{5}$$

and since η is often large compared to 1, the critical field in the ab-plane is generally much larger than the (already large!) one along the c-axis. For a field making an arbitrary angle θ with the ab-plane a straightforward calculation gives the result

$$H_{c2}(\theta) = \frac{H_{c2}^{\perp}\eta}{(\cos^2\theta + \eta^2\sin^2\theta)^{1/2}} \tag{6}$$

[3] In the literature, the conventional symbol for this ratio is γ, but I avoid this because of the possibility of confusion with $\gamma_{ab}(T)$.

Note that if $\eta \gg 1$, this means that for all but the smallest values of θ the critical field is determined by the condition $H_c(\theta)\sin\theta = H_{c2}^{\perp}$, i.e. the c-axis component of the field is equal to the critical field in this direction. The physical reason for this result is that almost all the energy of the vortices is associated with currents flowing in the ab-plane, very little with those flowing along the c-axis.

If one looks at the detailed expressions for ξ and λ, one sees that the $f(T)$ which occurs in eqn. (2) is given by[4]

$$f(T) = \left(\frac{\hbar}{2e^2}\frac{1}{2}\,|\alpha(T)||\Psi(T)|^2\right)^{1/2} = \left(\frac{\hbar}{2e^2}\frac{1}{4}\,\Delta F(T)\right)^{1/2} \tag{7}$$

where $\Delta F(T)$ is the condensation energy of the superfluid phase relative to that of the normal phase at the same temperature. Since this quantity can be measured directly in specific-heat experiments, the product $\xi_i(T)\lambda_i(T)$ can in principle be found for any given cuprate. If one could also measure $H_{c2}^{\perp}$ and $H_{c2}^{\parallel}$ accurately, one would be able to combine this result with eqns. (3, 4) and obtain accurate values for all four quantities ξ_{ab}, ξ_c, λ_{ab} and λ_c, In practice, it is difficult to implement this program because the large fluctuations that occur in the EM behavior of the high-temperature superconductors mean that the whole concept of an "upper critical field" is not very well defined experimentally, see below. One might think that an alternative procedure would be to measure the (eigenvalues of the) *lower* critical field $H_{c1}^{\parallel}, H_{c1}^{\perp}$ and relate them to $\lambda_{ab}(T)$ and $\lambda_c(T)$ by the standard argument, but in practice reproducible measurements of H_{c1} turn out to be notoriously difficult in the cuprates. Probably the most reliable technique for estimating the eigenvalues of ξ and λ in the cuprates is to combine eqn. (7) with direct (e.g. microwave or μSR) measurements of λ. Where this is done, one finds that the prefactor ξ_{ab} of the $(1-T/T_c)^{-1/2}$ in $\xi_{ab}(T)$ is comparable for YBCO (a-axis) and Bi-2212, and of order $15-25$Å. The c-axis values are however very different: for (optimally doped) YBCO the experimental value[5] of $\lambda_c(0)$ is $\sim 1.1\mu$; since $\lambda_a(0)$ $\sim$1600Å, this gives by (2) a ξ_c of the order of $2-3$Å (already smaller than the inter-bilayer spacing). For BSCO-2212 $\lambda_c(0)$ has the enormous value 100μ, so ξ_c would be of the order of 0.05 Å, making the range near T_c where the 3D GL theory is applicable very small.[6]

We now turn to the question of how to proceed when the 3D GL description fails because $\xi_c(T)$ becomes $\lesssim$ the inter-multilayer spacing. So far we have avoided the question of "where" in the unit cell the superconductivity is located (nothing has depended on this, since we have used a continuum description), but we must now face up to it. So we raise the question: do we know *for sure* that superconductivity in the cuprates is primarily associated with the CuO_2 planes? Certainly this seems the overwhelmingly

[4]This relation is often written (cf. Tinkham eqn. 9.6) in terms of the (isotropic) thermodynamic critical field $H_c(T)$, but the latter is not independently measurable for a type-II superconductor.

[5]Bonn et al., in G V. Note that what is quoted is the zero-temperature penetration depth $\lambda(0)$ rather than the prefactor λ_0 of $(1-T/T_c)^{-1/2}$ in $\lambda(T)$; the *anisotropy* of the two quantities is only slightly different, see Bonn et al., Fig. 18.

[6]Tinkham (p. 321) quotes values of $\gamma(\equiv$ our $\eta)$ of 7 for YBCO and 150 for Bi-2212. The former is consistent with the above estimate but the latter smaller by a factor ~ 3.

natural assumption, since at first sight these planes are the only element the various superconducting cuprates have in common. J. Dow has challenged this view, pointing out that another element that may be common to all superconducting cuprates is the "charge reservoir" layers.[7] It is of course clear that these layers, as their name implies, play an essential role in high-temperature superconductivity, at least as donors of holes (or electrons); but if the claim is that they are conducting and the Cooper pairs (?) form primarily in them rather than in the CuO_2 planes, it would seem to stretch credulity that the chemically and structurally very different charge reservoir layers found in (e.g.) YBCO, Tl-2201, Bi-22l2 and Hg-1201 nevertheless give rise to such similar T_c's and (at least qualitatively) similar behavior in the superconducting state.[8] This is not to deny that there are important questions concerning these layers, not the least of which is the extent to which they are metallic; it is entirely possible that if they are, Cooper pairing is induced in them by a proximity-type effect from the CuO_2 planes. If so, this may affect the quantitative details of the description, of the c-axis properties, but I shall assume it does not affect the general scheme to be described. Thus, I shall take it as a given from now on that the *principal* seat of superconductivity is indeed the CuO_2 planes.

Let's start by considering a "single-plane" material such as Tl-2201. Then it is natural to define a discrete order parameter $\Psi_n(\mathbf{r}_\parallel)$ for the n-th CuO_2 plane, where $\mathbf{r}_\parallel$ is the in-plane (xy-) coordinate. As a function of this order parameter the GL free energy would be expected to have the usual "bulk" terms, and the terms corresponding to inplane bending should also be essentially identical to those for the 3D case. However, we should expect that for "bending" along the c-axis the continuous gradient terms would be replaced by an expression proportional to the square of the difference of the discrete quantities $\Psi_n(\mathbf{r}_\parallel)$ and $\Psi_{n+1}(\mathbf{r}_\parallel)$. The lowest-order expression that is compatible with gauge invariance, etc., is

$$\begin{aligned} F^{(n)}_{\rm bend}(\mathbf{r}_\parallel) &= |\Psi_{n+1}(\mathbf{r}_\parallel) - \Psi_n(\mathbf{r}_\parallel)|^2 \\ \equiv |\Psi_{n+1}(\mathbf{r}_\parallel)|^2 + |\Psi_n(\mathbf{r}_\parallel)|^2 &- 2|\Psi_n(\mathbf{r}_\parallel) \parallel \Psi_{n+1}(\mathbf{r})| \cos\Delta\varphi_n(\mathbf{r}_\parallel) \end{aligned} \tag{8}$$

where $\Delta\varphi_n(\mathbf{r}_\parallel) = \varphi_{n+1}(\mathbf{r}_\parallel) - \varphi_n(\mathbf{r}_\parallel)$, φ_n being the phase of the complex quantity Ψ_n. Thus, this term has the characteristic form of a *Josephson coupling* between neighboring CuO_2 planes.

The total free energy obtained in this way has the form

$$F = \sum_n \int d\mathbf{r}_\parallel F_n\{\Psi_n(\mathbf{r}_\parallel)\} \tag{9}$$

where

$$\begin{aligned} F_n\{\Psi_n(\mathbf{r}_\parallel)\} \equiv -\alpha(T)|\Psi_n(\mathbf{r}_\parallel)|^2 + \frac{1}{2}\beta(T)|\Psi_n(\mathbf{r}_\parallel)|^4 \\ +\gamma_\parallel(T)|(\boldsymbol{\nabla}_\perp - 2ie\,A_\parallel(\mathbf{r})/\hbar)\Psi_n(\mathbf{r}_\parallel)|^2 \\ +K|\Psi_{n+1}(\mathbf{r}_\parallel) - \Psi_n(\mathbf{r}_\parallel)|^2 \end{aligned} \tag{10}$$

[7] At first sight the "infinite-layer" system $Ca_{1-x}Sr_xCuO_2$ has no charge reservoir layers, but the question is complicated by uncertainties as to whether this material is single-phase.

[8] cf. also the existence of the "infinite-layer" compound, which has no charge-reservoir component.

Here we chose for simplicity a gauge in which $A_z(\mathbf{r}) \equiv 0$ (this is always possible). If for any reason we wish to deviate from this condition, then to maintain gauge invariance we should replace the last term, most generally, by an expression of the form[9]

$$K\left|\Psi_{n+1}\exp\left(i\int_{z_n}^{z_{n+1}} 2e\mathbf{A}\cdot d\boldsymbol{l}/\hbar\right) - \Psi_n\right|^2 \tag{11}$$

The model described by eqn. (10) is known as the *Lawrence-Doniach model*; we see that it is equivalent to regarding the CuO_2 planes as a set of Josephson junctions in series. An important point to notice is that while it would be formally possible to incorporate the terms in $|\Psi_n(\mathbf{r}_\parallel)|^2$ arising from the last (K-) term in the term $-\alpha(T)|\Psi(\mathbf{r}_\parallel)|^2$, *this is not a natural thing to do*; we expect physically that in equilibrium $\Psi_{n+1} = \Psi_n$ and thus the last term as a whole is zero, so that the equilibrium value $\Psi(T)$ of Ψ_n is (in the mean-field approximation) the same as in a 3D case with the same parameters α and β. Thus, in any simple model of the LD type, *we do not expect the interplane Josephson coupling to raise* T_c*!*[10]

It is clear that under conditions where $\Delta\varphi_n$ is small compared to 1, i.e. where the bending of the order parameter over the interplane distance d is small, the LD model reduces to a GL theory, with the correspondence

$$\gamma_\perp = Kd. \tag{12}$$

We can also relate K to the critical current density J_c of the individual junctions (which under appropriate conditions is the c-axis critical current density of the material as a whole) by considering the case where the system is in equilibrium apart from the c-axis phase bending (so that $|\Psi_n(\mathbf{r}_\parallel)| = \text{const.} = \Psi(T)|$). We then get, using the standard form $(-I_c\Phi_0/2\pi)\cos\Delta\varphi$ for the Josephson coupling, the relation

$$K = J_c\Phi_0/(4\pi|\Psi(T)|^2) \tag{13}$$

and so

$$J_c = (4\pi/\Phi_0)\gamma_\perp|\Psi(T)|^2/d \tag{14}$$

Finally, noting that the c-axis penetration depth λ_c is given in terms of $\gamma_\perp$ by the expression

$$\lambda_c^{-2} = \frac{8e^2}{\hbar^2}\mu_0\gamma|\psi|^2 \tag{15}$$

we obtain a relation between λ_c and J_c of the form (SI units)

$$J_c = \hbar\epsilon_0 c^2/(2ed\cdot\lambda_c^2) \tag{16}$$

[9] Tinkham's eqn. (3) is consistent with this in the limit that A_z is slowly varying over the interlayer spacing.

[10] This point has been widely misunderstood in the literature, where one can find numerous attempts to invoke this coupling to explain the increase of T_c with n in homologous series. Of course, once one goes beyond the standard mean-field approximation, the situation becomes more complicated; cf. also lecture 13.

Thus, for example, for Bi-2212 where λ_c at low T has been measured at low temperatures to be $\sim 100\mu$, we infer[11] $J_c \sim 2\text{kA/cm}^2$. It should be emphasized that the relation (14) between J_c and $\gamma_\perp$ (and hence also eqn. (16)) applies only in the "true LD limit" where $\xi_c(T) \ll d$; in the opposite case, the d in eqn. (14) is replaced, up to a factor ~ 1, by $\xi_c(T)$. Thus for example, we expect the critical current to vary as $(1 - T/T_c)$ for T not too close to T_c but to crossover eventually to a $(1 - T/T_c)^{3/2}$ behavior as T approaches T_c and we get into the "3D GL" regime. It seems that for YBCO this regime is already reached at $1 - T/T_c \sim 0.1$, while for the much more anisotropic BSSCO-2212 compound it occurs only at $1 - T/T_c \sim 10^{-3}$ and hence is barely visible: BSCCO is almost always in the "true LD" limit.

One may ask whether, apart from its a priori plausibility, there is any direct experimental evidence for the picture of the CuO_2 planes as a series of Josephson junctions in series? If this view is correct, then one would expect that under appropriate conditions the *nonlinear* current-voltage characteristics would show the typical Josephson features, and indeed this seems to be the case[12] in Bi-2212. Note that in the true LD limit, in strong distinction to the GL case, the critical current can be exceeded without heating the sample into the normal phase.

So far, we assumed we are dealing with a single-plane material. What about multiplane materials such as Bi-2212? The most obvious assumption (which I have implicitly used a couple of times above, when referring to experimental data on this compound) is that the CuO_2 planes within a single multilayer are coupled together as strongly that it is legitimate in the present context to treat each multilayer as a single plane; then the above analysis goes though unchanged. However, it is not in fact quite certain that the coupling within a given multilayer is much stronger than that between different multilayers, and there are even a few pieces of evidence (e.g. the fact that the λ_c of Hg-1223, as inferred by Panagopoulos et al. from powder magnetization measurements,[13] is a factor of 5 *larger* than that of Hg-1201) which might suggest the opposite conclusion. I would regard this question as currently open: if indeed the "unexpected" result is correct, a quantitatively correct account of the bilayer cuprates would require the appropriate (and obvious) generalization of the LD model (the 3D GL model is, of course, insensitive to this complication).

A great deal of experimental work on the static magnetic properties of the cuprates has been done and interpreted in terms of the LD theory (or its limiting form, the 3D GL theory). A particularly interesting situation occurs when the external magnetic field is neither parallel nor perpendicular to the ab-plane. Under these conditions one expects to produce vortices that are *on average* parallel to the field. However, it is easily verified that to produce a given *current* in the c-direction costs an energy $\gamma_\parallel/\gamma_\perp$ times that necessary to produce the same current in the ab-plane ($E \sim J^2/\rho_s \sim J^2/\gamma$!), and thus the currents much prefer to flow in the planes. The result is a set of so-called "pancake" vortices that are staggered from one plane to the next, and the magnetization is not

[11] Assuming that a bilayer can be treated in this context as equivalent to a single plane, cf. below.
[12] Kleiner and Milller, Phys. Rev. B **49**, 1327 (1994).
[13] However, note the caveats in lecture 7 on this technique.

parallel to the field but oriented more along the c-axis: see Tinkham, Section 9.3.

Let's now very briefly turn to the question of the resistivity of the cuprate superconductors in a magnetic field. This could itself easily be the subject of a whole course: here I have space only for the barest essentials. The two basic qualitative points to appreciate are (1) that "superconductivity" in the sense of zero resistivity cannot be maintained in the presence of vortices, unless these are pinned, and (2) that because of the very different orders of magnitude of the relevant parameters, in particular temperature, it is far more difficult to pin vortices than in a classic superconductor. As a result, the question "are cuprate superconductors in a magnetic field really superconducting?" does not have a trivial answer. To take point (1) first, a vortex of circulation $\hat{\mathbf{n}} \oint \mathbf{v}_s \cdot d\boldsymbol{l} = \boldsymbol{\kappa}$ ($\hat{\mathbf{n}}$ = direction of axis) placed in a flow field such that the flow velocity at ∞ is $\mathbf{v}$ will find a so-called Magnus force of magnitude

$$\mathbf{F}_{\mathrm{M}} = \rho \mathbf{v} \times \boldsymbol{\kappa} \tag{17}$$

where ρ is the density of the fluid forming the vortex. The Magnus force has nothing to do with quantum mechanics (it was originally discovered in *classical* fluids); for a neutral system it is straightforward to obtain it by considering a tube of finite width and calculating the total kinetic energy as a function of vortex position (for a charged system where the vortex is effectively of finite extent, $\sim \lambda$, this argument in its simple form does not work, but more sophisticated arguments give the same result (cf. Tinkham, Section 5.2)). In the case of a superconductor, $\boldsymbol{\kappa}$ is equal to $\hat{\mathbf{n}}\,(h/2m)$, and if we assume that the bulk velocity $\mathbf{u}$ is associated with the same "density" ρ as appears in (16) (i.e. the superfluid density ρ_s) then we can rewrite (16) in terms of the electric current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{F}_{\mathrm{M}} = (\mathbf{J} \times \hat{\mathbf{n}})\Phi_0 \qquad\qquad (\Phi_0 \equiv h/2e) \tag{18}$$

Note that this relation is independent of the value of ρ_s and hence of T.

If the Magnus force $\mathbf{F}_{\mathrm{M}}$ is not balanced by some "pinning" force that tends to keep the vortex close to a given impurity (etc.), then its effect will be to accelerate the vortex transverse to the current $\mathbf{J}$; eventually its effect will be balanced by some frictional force, and the vortex will reach a terminal (steady-state) velocity $\mathbf{u}$, which is the simplest case would be expected to be proportional to $\mathbf{v}$.

Now, consider the total phase difference $\Delta\varphi_{12} \equiv \int_1^2 \boldsymbol{\nabla}\varphi \cdot d\boldsymbol{l}$ between two points in the system separated along the direction of current flow: for definiteness we choose a straight contour to connect them. Whenever a vortex moves across the contour, the integral decreases by an amount 2π. But according to the Josephson relation, we expect the voltage difference V_{12} between the points 1 and 2 to be proportional to the rate of charge of $\Delta\varphi_{12}$:

$$\frac{d}{dt}\Delta\varphi_{12} = \frac{2e}{\hbar} V_{12} \tag{19}$$

Consequently, the *average* voltage $\overline{V}_{12}$ is

$$\overline{V}_{12} = \frac{\hbar}{2e}\,\frac{d}{dt}\Delta\varphi_{12} = \frac{\hbar}{2e} 2\pi\; n_v u s_{12} \equiv \Phi_0 n_v u s_{12} \tag{20}$$

where n_v is the number of vortices per unit area and s_{12} the distance between 1 and 2. Thus if u is linearly proportional to $\mathbf{J}$ as in the simplest case, the ratio of V_{12} to J is proportional to s_{12} and to $\mathbf{J}$, i.e. the system displays a simple ohmic resistive behavior. (In the more general case, the I-V characteristic is nonlinear.). This is known as the *flux-flow mechanism of resistivity*. To calculate an actual value for the resistivity ρ, one needs a theory of the frictional force acting on a moving vortex: the simplest (Bardeen-Stephen) theory yields a linear friction coefficient $\eta \approx \Phi_0 H_{c2}/\rho_n$ where ρ_n is the normal-state resistivity, and this then gives the remarkable result that in the limit $H \to H_{c2}$ the flux-flow resistivity approaches the normal-state value.

The above discussion rested essentially on the assumption that the vortices are free to move. In real life, they tend to be pinned by impurities, lattice defects etc., which is why many classic type-II superconductors have essentially zero resistivity even in fields comparable to H_{c2}. As a result of the pinning plus the Magnus force, the effective potential seen by a vortex is of the form shown; at zero temperature the vortex will be pinned in the metastable well, but at finite temperature there will be the chance of thermal activation over the barrier, with a rate proportional to the Arrhenius factor $\exp -V_0/k_{\mathrm{B}}T$ (note that V_0 is a strongly decreasing function of increasing J). Now it turns out (not obviously!) that typical (single-defect) pinning energies in the cuprates are not very different in order of magnitude from those in classic superconductors, while T_c is much larger; consequently, it is very difficult to avoid depinning. The situation is further complicated by the fact that at lower temperatures interactions between vortices may lead to collective pinning ("glass transition"): see Tinkham, Section 9.5.