Lecture 3. Phenomenological theory of the EM properties of superconductors*

1. London theory (F. and H. London, 1935) [Recap on significance of Meissner effect]
   Consider first T=0, assume all electrons behave in "superconducting" way. Eqn. of motion in normal metal would be

$$\frac{d\boldsymbol{J}}{dt} = \frac{ne^2}{m}\boldsymbol{E} - \frac{\boldsymbol{J}}{\tau} \Longrightarrow \sigma = \frac{ne^2\tau}{m}$$

Experimentally $\sigma\to\infty$, so $1/\tau\to 0$. Thus eqn. of motion of electrons in superconductor is

$$\frac{d\boldsymbol{J}}{dt} = \frac{ne^2}{m}\boldsymbol{E}$$

$$\Rightarrow \frac{d}{dt}\nabla\times\boldsymbol{J} = \boldsymbol{\nabla}\times\frac{d\boldsymbol{J}}{dt} = \frac{ne^2}{m}\boldsymbol{\nabla}\times\boldsymbol{E} \underset{\text{Maxwell}}{=} \frac{ne^2}{m}\left(\frac{-\partial\boldsymbol{B}}{\partial t}\right)$$

i.e.

$$\frac{\partial}{\partial t}\left\{\boldsymbol{\nabla}\times\boldsymbol{J} + \frac{ne^2}{m}\boldsymbol{B}\right\} = 0$$

$$\Rightarrow \boxed{\boldsymbol{\nabla}\times\boldsymbol{J} + \frac{ne^2}{m}\boldsymbol{B} = const.} \quad \text{(in time)}$$

So far, nothing new – above simply a consequence of infinite conductivity.
[in particular, $\Phi+(m/ne^2)\oint \boldsymbol{J}\cdot d\boldsymbol{l}$ = const. – (Lippmann's rule)]

But: Meissner shows $\boldsymbol{B}$=0 in interior of superconductor. So, Londons postulate that the const.=0, i.e.

$$\boxed{\boldsymbol{\nabla}\times\boldsymbol{J} + \frac{ne^2}{m}\boldsymbol{B} = \boldsymbol{0}} \quad \text{- London eqn.} \qquad (*)$$

Combine with Maxwell eqn. $\nabla\times\boldsymbol{H} = \boldsymbol{J} + \partial D/\partial t \leftarrow$ zero if t – independent situation
$\Rightarrow$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \frac{-ne^2}{m}\mu_o\boldsymbol{B}$$

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* Refs: F. London, Superfluids, Tinkham ch. 1, Rickayzen. Note, historically this material is all pre-BCS.

or since $\nabla \cdot \boldsymbol{B} \equiv 0$, (and $\nabla \cdot \boldsymbol{J} \equiv 0$ in time-independent situation)

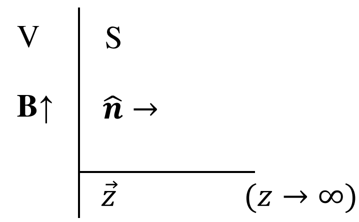
$$\nabla^2 \boldsymbol{B} = \lambda_L^{-2} \boldsymbol{B} \text{ and } \nabla^2 \boldsymbol{J} = \lambda_L^{-2} \boldsymbol{J}$$

with $\lambda_L^2 \equiv \frac{m}{ne^2}\left(= \left(\frac{c^2}{\omega_p^2 \epsilon}\right)\right) \sim (10^{-5}\ cm)^2 -$ London penetration depth (de Haas-Lorentz)

Note λ_L is just HF skin depth in N phase, but now has quite different significance: e.g. infinite flat-plate geometry:

$$B(z) = B_o exp - \lambda_L^{-1} z$$

$$\boldsymbol{J} = \mu_o^{-1} \boldsymbol{\nabla} \times \boldsymbol{B} = \hat{\boldsymbol{t}} \mu_o^{-1} \lambda_L exp - \lambda_L^{-1} z \qquad (\hat{t} \text{ into page})$$



Screening currents on surface, B screened out in $o(\lambda_L)$.

Note: at surface of a superconductor occupying an infinite half-space, $\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0$, i.e. magnetic field is parallel to surface. Proof by reduction ad absurdum: if, $\hat{\boldsymbol{n}} \cdot \boldsymbol{B} \neq 0$ i.e. $B_z \neq 0$ just inside superconductor, then from div $\boldsymbol{B} = 0$ and translation invariance in parallel direction, $B_z \neq 0$ infinitely far into superconductor. But then by London eqn. $\partial J_x/\partial y$ and/or $\partial J_y/\partial x \neq 0$, violating condition of TI ‖ to surface. For a finite geometry, this argument suggests that $\boldsymbol{B}$ is approximately parallel to surface provided all dimensions are $\gg \lambda_L$ (e.g. macroscopic sphere).

For samples with one or more dimensions $\lesssim \lambda_L$, situation more complicated: e.g. for infinite thin plate, $d \lesssim \lambda_L$, effective no. of electrons (n) reduced by factor ~ d/λ where λ is "effective" 2D penetration depth. Thus, $\lambda^2 \sim \lambda_L^2 \left(\frac{\lambda}{d}\right) \Rightarrow \lambda_{2D} \sim \frac{\lambda_L^2}{d}$. Note that for a "2D" slab the current does <u>not</u> flow principally around boundaries but through bulk!

Finite T: $n_s(T)$ of e^-'s superconducting, $n_n(T) \equiv n - n_s(T)$ "normal". At dc, normal e^-'s don't contribute $\Rightarrow$ formula same except

$$\lambda_L^2(T) = \frac{m}{n_s(T) c^2 \mu_o} \equiv \frac{n}{n_s(T)} \cdot \lambda_L^2(0)$$

Assume $n_s \rightarrow n$ at T=0, and $\rightarrow 0$ at $T \rightarrow T_c$, then $\lambda_L(T) \rightarrow \infty$ as $T \rightarrow T_c$. If we make "default" assumption $n_s(T) \sim T_c - T$ for $T \rightarrow T_c$, then $\lambda_L(T) \sim (T_c - T)^{-1/2}$. (Approximate empirical relation: $\lambda^2(T) \sim \lambda^2(0)(1 - (T/T_c)^4)^{-1/2}$)

Experimental measurement of λ: inductance of cavity, colloid suspensions Josephson effect… Note that generally it is easier to measure <u>changes</u> in λ with some parameter (e.g. T) than absolute value.

2 Implications of London eqn.

Since $\boldsymbol{B} \equiv \nabla\times\boldsymbol{A}$, the London equation[(*)] can be rewritten

$$\boldsymbol{\nabla}\times\left(\boldsymbol{J}+\frac{ne^2}{m}\boldsymbol{A}\right)=0$$

i.e. $\quad \boldsymbol{J}+\frac{ne^2}{m}\boldsymbol{A}=\nabla\psi(r)$

In simply-connected sample, in region beyond λ_L where $\boldsymbol{J}=0$, $\boldsymbol{A}=\nabla\psi \Rightarrow \nabla\times\boldsymbol{A}\ (=\boldsymbol{B})=0 \Rightarrow \boldsymbol{A}$ = longitudinal & can be gauged away. Hence in any simply-connected "large" sample, can write for <u>all</u> $\mathbf{r}$,

$$\boldsymbol{J}^{(r)}=\frac{-ne^2}{m}\boldsymbol{A}(\boldsymbol{r})$$

But in longitudinal case we know $\boldsymbol{A}$ can induce no $\mathbf{J} \Rightarrow$ system knows difference" between L and T forms of A even in limit $\mathbf{q}\to 0$.

Perturbation theory: in presence of $\boldsymbol{A}$.

$\boldsymbol{p}_i \to \boldsymbol{p}_i - e\boldsymbol{A}_i(\boldsymbol{r})$

$$\Rightarrow \mathcal{H}'=-\sum_i e\boldsymbol{p}_i\frac{\boldsymbol{A}}{m}(r_i)+\sum_i e\frac{\boldsymbol{A}^2}{2m}(r_i)$$

But current

$$\boldsymbol{j}(\boldsymbol{r})=\frac{e}{m}\sum_i\left(\boldsymbol{p}_i-e\boldsymbol{A}(\boldsymbol{r}_i)\right)$$

$$\Rightarrow\quad \frac{\boldsymbol{\delta J}(r)}{\boldsymbol{\delta A}(r')}=\sum_n\frac{\langle 0|\boldsymbol{J}(\boldsymbol{r})|n\rangle\langle n|\boldsymbol{J}(\boldsymbol{r}')|0\rangle}{E_n-E_o}-\frac{ne^2}{m}\delta(\boldsymbol{r}-\boldsymbol{r}')$$

take F.T.:

$$\frac{\delta J_k}{\delta A_k}=\sum_n\frac{|\langle 0|J_k|n\rangle|}{E_n-E_o}-\frac{ne^2}{m}$$

For L case, f-sum rule ensures $\delta J_k/\delta A_k=0$ as above (no response to purely longitudinal static vector potential). For T case, get London result if we assume that for some reason matrix elements of $J_k\to 0$ with k but (relevant) energy levels stray finite ("rigidity", gap). Cf. atomic diamagnetism (Bohr-van Leeuwen theorem)

In multiply connected case (e.g. ring) cannot necessarily infer $\boldsymbol{A} = 0$ in middle of ring $\Rightarrow$ possibility of trapped flux. (but no statement about what values possible, for now)

Analogy between Meissner diamagnetism and HF effect in superfluid ^4He:

If we place a normal liquid (including ^4He above T_λ) is an annular container and rotate the container slowly $\left(\omega < \omega_e \equiv \frac{1}{2}\hbar/mR^2\right)$, the liquid rotates with the container. If now in the case of ^4He we cool through T_λ and on down towards $T = 0$, the liquid comes out of equilibrium with the container and as $T\to 0$, is (approximately) at rest in the lab frame. This is the Hess-Fairbank (HF) effect (or nonclassical rotational inertia, NCRI).

To see the correspondence with Meissner diamagnetism, consider the Hamiltonian formulation of the problem in the rotating frame. (indicate variables in this frame by primes). For a single particle the canonical momentum $\boldsymbol{p}'$ is $m(\dot{r}' + \boldsymbol{\omega} \times \boldsymbol{r}')(so\ \boldsymbol{j}' = m^{-1}(\boldsymbol{p}' - m\boldsymbol{\omega} \times \boldsymbol{r}'))$, and the (canonical) Hamiltonian is

$$H'(\boldsymbol{r}', \boldsymbol{p}') = \frac{(\boldsymbol{p}' - m\boldsymbol{\omega} \times \boldsymbol{r}')^2}{2m} + \tilde{V}(\boldsymbol{r}')$$

$$\tilde{V}(\boldsymbol{r}') = V(\boldsymbol{r}') - \frac{1}{2}m(\boldsymbol{\omega} \times \boldsymbol{r}')^2 \quad \leftarrow \text{centrifugal term}$$

and $\boldsymbol{j}' = (\boldsymbol{p}' - m\boldsymbol{\omega} \times \boldsymbol{r}')/m$

Compare case of electrically charged system, viewed from lab. frame but in presence of EM vector potential $\boldsymbol{A}(\boldsymbol{r})$:

$$H(\boldsymbol{r}, \boldsymbol{p}) = \frac{\left(\boldsymbol{p} - e\boldsymbol{A}(r)\right)^2}{2m} + \mathrm{V}(r)$$

$$\boldsymbol{j} = (\boldsymbol{p} - e, \boldsymbol{A}(r))/m$$

except for centrifugal term, exact correspondence between EM system viewed from lab. frame & neutral system viewed from rotating frame, with $eA(\boldsymbol{r}) \leftrightarrows m(\boldsymbol{\omega} \times \boldsymbol{r})$, or for constant field $\boldsymbol{B}$ such that $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$, $\boldsymbol{\omega} \leftrightarrows e\boldsymbol{B}/2m$.

In particular, nonzero EM current in lab. frame $\leftrightarrows$ nonzero neutral-atom current in rotating frame.

[Can generalize straightforwardly to many-body case provided $\mathrm{V}(\boldsymbol{r}'_i - \boldsymbol{r}'_j) = \mathrm{V}(\mathbf{r}_i - \mathbf{r}_j)$]

3 Pippard modification.

There are two obvious problems with the London theory:

(1) it does not explain the possibility or nature of type-II superconductivity.

(2) The actual value of the experimental penetration depth, as measured e.g. from inductance experiments, is often considerably greater than the London value

$\sqrt{m/n_c(T)e^2\mu_o}$ and moreover is very sensitive to alloying, even though thermodynamic properties little affected.

Pippard hypothesis (discussed in much more detail later, in context of BCS theory): $\boldsymbol{J}(\boldsymbol{r})$ is a <u>nonlocal</u> function of $\boldsymbol{A}(\boldsymbol{r})$, i.e.* [pure material for now]

$$\boldsymbol{J}(r) \sim \int K(r,r')\boldsymbol{A}(r')dr' \qquad \text{(t)}$$

where range of K is of order some length ξ_o ("Pippard coherence length"). If $\xi_o \ll \lambda_L(T)$, then essentially reduces to London theory provided $\int K(rr')dr' = n_s(T)e^2/m$. What if $\xi_o \gtrsim \lambda_L(T)$? Suppose actual penetration depth is $\sim\lambda$. Then the contribution to the RHS of (t) is $\sim A(r) \times n_s(T)e^2/m \times (\lambda/\xi_o) = A(r) \cdot \lambda_L^{-2}(\lambda/\xi_o)$. Thus,

$$\lambda^{-2} \sim \lambda_L^{-2}\left(\frac{\lambda}{\xi}.\right)$$

$$\Longrightarrow \lambda \sim (\lambda_L^2\xi_o)^{1/3}, \text{ which can be } \gg \lambda_L.$$

In a <u>dirty</u> material (mfp $\ell \ll \lambda_{\text{pure}}$) then Pippard supposed reduction would be by a factor ℓ/ξ_o rather than (λ/ξ_o). Thus,

$$\lambda^{-2} \to \lambda_L^{-2}\,(\ell/\xi_o)$$

$$\Rightarrow \lambda \sim \lambda_L(\xi_o/\ell)^{1/2}$$

So in Pippard approach, ξ_o is essentially the range of nonlocality (in a pure metal) of electromagnetic effects. It turns out (from the experiments on $\lambda(T)$) that ξ_o is <u>only weakly sensitive to temp, and in particular does not diverge for $T \to T_c$</u>.: in hindsight, will interpret ξ_o as essentially radius of Cooper pairs.

Definition of London and Pippard limits: note always in London limit for (a) sufficient dirt, and (b) for $T \to T_c$ ($\lambda_L \to \infty$, $\xi_o \sim$ finite) (crucial for validity of GL approach). [Still no explanation of type-II…]

4 <u>GL theory: type-II superconductivity</u>

Suppose we apply an external field of the order of the thermodynamic critical field H_c to the sample. Let's consider the possibility that it punches holes (vortices) through, with a normal core (since $\mathbf{H} = 0$ in bulk S) and circulating currents around the core. Is this energetically advantageous? [for a more quantitative calculation, see Tinkham section 4.3., which follows the historical arguments more closely]. Consider first for definiteness T=0.

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* Specific choice: $K(rr') \sim \frac{\boldsymbol{R}\boldsymbol{R}'}{R^4} \exp - R\left\{\frac{1}{\xi_o} + \frac{1}{\ell}\right\}, \qquad R \equiv \boldsymbol{r} - \boldsymbol{r}'$ (Chambers)

First, what is the gain in energy? Essentially, we expect that the field punches through over a region of dimension ~λ, so the gain per unit length of vortex line is $\sim H_c^2\lambda^2$. On the other hand we need to form a normal core. Let's assume that the "bending energy" to go from S to N over a distance L is K/L^2 per unit vol., and define a length ξ so that $K = E_{\text{cond}} \cdot \xi^2$, where E_{cond} is the combination energy. Then the total bending energy per unit area is independent of L and of order $E_{\text{cond}}\xi^2 \sim H_c^2\xi^2$ (df of H_c!), while the loss of "bulk" condensation energy is $\sim E_{\text{cond}} L^2$: thus, for L $\lesssim$ξ this term is of the same order as the bending energy, and the total energy/unit length of the vortex line is given by

$$E \sim H_c^2(a\xi^2 - b\lambda^2) \qquad a,b \sim 1.$$

Thus, for $\xi \gg \lambda$ the energy is positive and it is not advantageous to introduce vortex lines, but for $\lambda \gg \xi$ it becomes advantageous.

[At finite T, $E \to G(T)$ but argument otherwise the same]. Note, so far, no specification of "strength" of vortex.

These considerations made more quantitative by phenomenological theory of GL (1950). Introduce complex "order parameter" (wave function) $\psi(r)$ and postulate followingexpression for free energy density (after Landau & Lifshitz):

$$F(\psi) = F_{no} + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m*}|\left(-i\hbar\boldsymbol{\nabla} - e^*\boldsymbol{A}(\boldsymbol{r})\right)^2\psi|^2 + \frac{1}{2}\mu_o^{-1}B^2$$

In this expression m* and e* are at present stage unknown, though it seems reasonable to guess they are ~ electron mass and charge. The coefficients α, β are given by

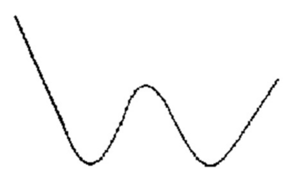
$$\alpha(T) = \alpha_o(T - T_c)$$

T > T$_c$

$$\beta(T) = \beta_o \ (\sim \textit{ind. of } T)$$

Thus for a <u>uniform</u> state, potential looks like:

The electric current is defined as ∂F/∂ A(**r**) and hence



T < T$_c$

$$J(r) = \frac{e*}{2m*}\left(\psi*\left(-i\hbar\boldsymbol{\nabla} - e^*\boldsymbol{A}(\boldsymbol{r})\right)\psi + c.c.\right)$$

just as for a single particle described by a Schrödinger wave function ψ. In the case where ψ is constant in space, $J(r) = -\frac{e*^2}{m*}|\psi|^2\boldsymbol{A}(r)$: thus, we can tentatively write

$$\lambda_L^{-2} = \frac{e^{*2}}{m^*}|\psi|^2.$$

Note *GL* implicitly assume a local response. (valid for $T \rightarrow T_c$, at least) If ψ is taken to be the equilibrium *OP,* it is given for $T < T_c$ by $|\psi|^2 = |\alpha|/\beta$ and thus $\propto T_c - T$; thus $\lambda_L \propto (T_c - T)^{-1/2}$ as observed.

The *GL* free energy defines another characteristic length which is independent of e^*, namely $\xi^2(T) = (\hbar^2/2m^*)|\alpha(T)|$. Since $\alpha(T) \sim T_c - T, \xi(T)$ also $\sim (T_c - T)^{-1/2}$. $\xi(T)$ is <u>*GL* coherence (correlation) length:</u> do not confuse with ξ_o!

The ratio of λ_L to ξ is independent of *T* for $T \rightarrow T_c$ and is usually denoted κ: from the above

$$\kappa = \frac{\hbar^2}{2}\left(\frac{e^*}{m^*}\right)^2 \frac{1}{\beta}$$

where β can be derived from the experimental values of $H_c(T)$ and $\lambda(T)$ (see Tinkham 4.1). Actually in BCS theory we have in the "clean" limit

$$\left.\begin{aligned} \lambda(T) &\sim \lambda(0)(1 - T/T_c)^{-1/2} \\ \xi(T) &\sim \xi_o\,(1 - T/T_c)^{-1/2} \end{aligned}\right\} \; T \rightarrow T_c$$

so κ is actually $\sim \lambda_L(0)/\xi_o$ (0.96 times this, in clean limit).

It follows from a detailed analysis (cf. l. 10) that the formation of vortices is favorable when $\kappa > 1/\sqrt{2}$. : thus this is the discriminant between type-I and type-II superconductivity. For a <u>clean</u> superconductor, the type-I – type-II distinction is essentially the same as Pippard-London. For a dirty superconductor, $\kappa \sim \lambda_L(0)/\ell$.

5. <u>The relevance of Bose condensation</u>

Consider simple neutral system of noninteracting particles (statistics so for unspecified) in narrow annular geometry, radius *R*. Container rotates at angular velocity **ω**.

$$\hat{H} = \hat{H}_o - \boldsymbol{\omega} \cdot \boldsymbol{L} = \sum_\ell n_\ell \underbrace{\left(\frac{\hbar^2 \ell^2}{mR^2} - \hbar\omega\ell\right)}_{\tilde{\varepsilon}_\ell} + E_{\text{transverse}} \leftarrow \text{drops out}$$

Expectation value of angular momentum:

$$\langle L \rangle = \sum_\ell n_\ell \; \hbar\ell.$$

normal state: classical, $n_\ell \sim N \exp -\beta\tilde{\varepsilon}_\ell \,/\sum_\ell ..$ (Fermi, Bose..) in all cases smoothly varying function: consider classical case (or F, B in nondegenerate regime) for simplicity.

$$\langle L\rangle = N\hbar \sum_\ell \ell \, exp - \beta\tilde{\varepsilon}_\ell / \sum_\ell exp \ \sim\beta\tilde{\varepsilon}_\ell$$

$$\equiv N\hbar \sum_\ell \ell \exp - \beta \left(\frac{\hbar^2\ell^2}{2mR^2} - \ell\hbar\omega\right) / \ \sum_\ell exp - ..$$

exponential function smooth for $\kappa T \gg \hbar^2/mR^2$; since smooth, $\cong N\hbar \int \ell \exp - \beta \ldots /$ $\int \exp - \beta \ldots$ Introduce new variables $\ell' \equiv \ell - \ell_o, \ell_o \equiv mR^2\omega/\hbar$ so as to complete square, then

$$\langle L\rangle = N\hbar \int d\ell'(\ell' + \ell_o) exp - \beta\left(\frac{\hbar^2\ell'^2}{2mR^2} - A\right) / \int d\ell' exp - \beta(\hbar^2\ell'^2/2mR^2 - A)$$

$$= N\hbar\, \ell_o \equiv NmR^2\,\omega \equiv I_{cl}\,\omega$$

ie <u>liquid rotates exactly with cylinder</u> (to $o\ (\hbar^2/\, m\, R^2kT \ll 1)$).

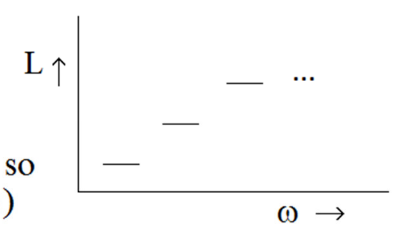
Now consider Bose system below T_c: "normal component" described by $n_k = (\exp\beta\ \epsilon_k \ldots 1)^{-1}$, <u>but</u> $\sum_k n_k \equiv N_n < N$, so define $N_o \equiv N - \sum_k n_k \equiv$ condensate no. (~N). These must all pile into the lowest single-particle state, i.e. one with minimum value of $\tilde{\varepsilon}_L$. Thus,

$$\langle L\rangle = N_{\boldsymbol{m}} R^2\omega + N_o\,\hbar\ell_o$$

$$\ell_o = \text{nearest integer to } \omega/\omega_c \ \ \omega_c \equiv \hbar/mR^2.$$

$\Rightarrow$

(T = 0, so $N_n = 0$)



6a <u>A possible definition of the "order parameter" for a BEC system:</u>

$$\Phi(rt) \equiv \sqrt{N_o}\chi_o(\boldsymbol{rt}) \qquad (N_0 = f(t) \text{ in general case})$$

Definition of characteristic velocity: in Schrödinger (single-particle) case

$$\rho(rt) = |\psi(rt)|^2$$

$$\boldsymbol{j}(rt) = -\frac{i\hbar}{2m}(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)$$

if introduce $\psi(rt) \equiv \Lambda(rt) \exp i\, \varphi(rt)$, then

$$\rho(\boldsymbol{r}t) = \Lambda^2(rt), j(\boldsymbol{r}t) = \frac{\hbar}{m}\Lambda^2\nabla\varphi(rt)$$

One can define "velocity" by

$$\mathbf{v}(\boldsymbol{r}t) \equiv \boldsymbol{j}(rt/\rho(rt) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi(rt)$$

"quantum" object, but not terribly useful physically, because subject to large fluctuations.

In BEC case, try defining.

$$\Psi(rt) = \Lambda(rt)\exp i\varphi(rt) \qquad (\text{or}\sqrt{N_o}\times \text{this, doesn't matter})$$

$$\mathbf{v}_{\boldsymbol{s}}(\boldsymbol{r}t) \equiv \frac{\hbar}{m}\boldsymbol{\nabla}\varphi(rt) \qquad \leftarrow \text{"superfluid velocity"}$$

satisfies: (a) curl $\mathbf{v}_s = 0$

(b) $\oint \mathbf{v}_s \cdot \boldsymbol{dl} = nh/m$ (Onsager-Feynman)

Note these conditions are <u>not</u> satisfied by "hydrodynamic" velocity of normal fluid

$$\left(\mathbf{v}_h(rt) \equiv \sum_i n_i A_i^2(\boldsymbol{rt})\nabla\varphi_i(rt)/\sum_i n_i A_i^2(\boldsymbol{rt})\right)$$

Thus, $\mathbf{v}_s$ is "quantum" object, but not subject to longer fluctuations because made up of contributions of $N_0 \sim N$ particles.

Charged system: p→p-$e\boldsymbol{A}$) so

$$\mathbf{v}_s = \frac{\hbar}{m}(\boldsymbol{\nabla}\varphi - e\boldsymbol{A}/\hbar)$$

$$\Rightarrow \oint \mathbf{v}_s \cdot d\boldsymbol{l} = \frac{\hbar}{m}(n - \Phi/(h/e))$$

and in particular if $\mathbf{v}_s = 0$ (e.g. in interior of thick ring) then

$$\Phi = nh/e \qquad (\text{London, with } e = \text{actual electron charge})$$