Problem Sheet 3

1. 'Generalized GL approach' at $T = 0$.

Consider the standard model of a BCS superconductor with an interaction $V(\mathbf{r}) \equiv -g\delta(\mathbf{r})$ ($g > 0$) and an energy cutoff $\epsilon_c \gg k_B T_c$, and take the limit of a homogeneous system at $T \to 0$. Define an 'order parameter' Ψ in the standard way, and define the 'gap' (= x-component of effective field in the Anderson pseudospin model) necessary to produce that Ψ by the implicit equation

$$\Psi(\Delta) = \sum_{\mathbf{k}} \Delta/2E_{\mathbf{k}}(\Delta), \quad E_{\mathbf{k}}(\Delta) \equiv \left(\epsilon_{\mathbf{k}}^2 + |\Delta|^2\right)^{1/2} \tag{1}$$

(a) Find explicit expressions for the potential and kinetic energies as functions of Δ. (In the latter case, you may find it easiest to use the argument of lecture 10 and perform an integration by parts.) Hence, show that provided that we consider only values of $\Delta \ll \epsilon_c$, the total energy relative to the normal ground state has the form

$$E(\Delta) = N(0)\Delta^2\left\{-g' \ln^2(2\epsilon_c/\Delta) + \ln(2\epsilon_c/\Delta) - 1/2\right\} \tag{2}$$

(where $g' \equiv gN(0)$).

(b) Introducing the notation $\Delta_0 \equiv 2\epsilon_c \exp -1/g'$ and using it to rewrite the expression for $E(\Delta)$, show that minimization of $E(\Delta)$ with respect to Δ leads to the BCS gap equation (i.e., to $\Delta = \Delta_0$), and that the corresponding condensation energy has the BCS value $-\frac{1}{2}N(0)\Delta_0^2$.

(c) Now consider small deviations of Δ from the BCS value Δ_0, $\Delta = \Delta_0 + \delta\Delta$. Find the coefficient of $(\delta\Delta)^2$ in $E(\Delta)$ and compare it with the coefficient of $\big(\delta\Delta(T)\big)^2$ extrapolated from the GL regime near T_c.

(d) Find the maximum value (in terms of Δ_0) of Δ at which the superconducting state is stable with respect to the normal state at $T = 0$.

(Note: in parts (b)–(d) it is necessary to bear in mind that $\epsilon_c \gg \Delta_0$ implies $g' \ll 1$ and hence values of Δ/Δ_0 of order $\exp g^{-1}$ or greater are unphysical.)

2. 'Critical fluctuations' in the GL theory.

Consider the GL free energy functional of a 3D system in the usual notation, where the coefficients of the three terms are $\alpha \equiv -\alpha_0 t$, β_0, γ_0, with $t \equiv 1 - T/T_c$.

(a) From dimensional considerations, or otherwise, show that the only dimensionless combination which can be formed from parameters α, β_0, γ_0 and the thermal energy $k_{\mathrm{B}}T_c$ is of the form $\left(\alpha^{1/2}\gamma_0^{3/2}/k_{\mathrm{B}}T_c\beta_0\right) \equiv \eta$. From this, deduce that if there is any value t_c of t such that the behavior changes qualitatively, it must be of order $\left(k_{\mathrm{B}}T_c\right)^2\beta_0^2/\alpha_0\gamma_0^3$.

(b) Consider the system at some (small) positive value of t, i.e., in the superconducting phase. Show that for $t \gg t_c$ the volume of the largest subvolume which can fluctuate thermally into the normal phase is determined primarily by the γ-term and is $\ll$ the healing length $\xi(T)$, but for $t \lesssim t_c$ is $\gtrsim \xi(T)$ and determined largely by the β-term.

(c) Using the 'clean BCS' values of α_0, β_0 and γ_0, show that t_c is of order $(k_{\mathrm{B}}T_c/\epsilon_{\mathrm{F}})^4$ and thus unobservably small in a typical 'classic' superconductor. How short would the mean free path l have to be for Al ($\xi_0 \approx 10^4$Å) before t_c became observably large (say 10^{-6})?

(d) Repeat the above arguments for a 2D system.

[In the general theory of second-order phase transitions, t_c corresponds (within a numerical factor) with the border of the 'critical regime' where standard mean-field theory fails qualitatively. The above argument is a 'poor man's version' of a famous argument originally due to Ginzburg.]

3. Andreev reflection.

In a given metal, the gap Δ is a function only of z; it tends to zero as $z \to -\infty$ and to a finite value Δ_0 as $z \to +\infty$. An electron is incident from $z = -\infty$ (normal metal side) with momentum at an angle θ ($< \pi/2$) relative to the positive z-axis and energy less than (but of order of) Δ_0. Assuming it undergoes Andreev reflection,

(a) Which of the following quantities characterizing the normal part of the system remain unchanged to lowest order in $\Delta/\epsilon_{\mathrm{F}}$: total particle number, spin, energy,* z-component of momentum, transverse component of momentum, electric current, thermal current? If they are changed, by how much? Can the condensate have absorbed the deficit?

(b) Find the change in momentum up to first order in $\Delta/\epsilon_{\mathrm{F}}$, as a function of θ and the energy of $\epsilon_{\mathbf{k}}$ of the incident electron, relative to the Fermi energy. Hence, assuming a simple Sommerfeld model of the normal state, find the direction of propagation of reflected wave packet up to order $\Delta/\epsilon_{\mathrm{F}}$.

(c) Now consider an electron incident with $\theta = 0$ but energy $\epsilon > \Delta_0$. Suppose the form of the gap as a function of z is

$$\Delta(z) = \frac{1}{2}\,\Delta_0(1 + \tanh(z/L)) \tag{3}$$

Find the reflection coefficient up to lowest order in Δ/ϵ_F in the cases

(i) "abrupt" N-S interface, $L \to 0$ (but† $L \gg k_F^{-1}$)

(ii) "infinitely graded" interface, $L \to \infty$

(Hint: in part (i) match the "particle" and "hole" components of the excitation wave function. In part (ii), use the "pseudospin" analogy).

(d) (optional, for extra credit): Referring to part (b), does this mean that electrons incident at grazing incidence transfer arbitrary large momentum? Why (not)?

(Note: It is helpful to remember that in the language of Bogoliubov quasiparticles, a quasiparticle of 'momentum $\mathbf{p}$' in the *normal* phase is for $|\mathbf{p}| > p_{\mathrm{F}}$ an electron with momentum $\mathbf{p}$, but for $|\mathbf{p}| < p_{\mathrm{F}}$ the absence of an electron with momentum $-\mathbf{p}$.)

Solutions to be put in 598SC homework box (2nd floor Loomis) by 1 p.m. on Mon. 10 Oct.

* relative to μ

† to avoid having to worry about "normal" reflection processes