Statics - TAM 211

Lecture 32

April 9, 2018

Chap 10.1, 10.2, 10.4, 10.8

Announcements

- **No class Wednesday April 11**
- **No office hours for Prof. H-W on Wednesday April 11**

- Upcoming deadlines:
  - Tuesday (4/10)
    - PL HW 12
  - Thursday (4/12)
    - WA 5 due
  - Monday (4/16)
    - Mastering Engineering Tutorial 14

Chapter 10: Moments of Inertia

Goals and Objectives

- Understand the term "moment" as used in this chapter

- Determine and know the differences between
  - First/second moment of area
  - Moment of inertia for an area
  - Polar moment of inertia
  - Mass moment of inertia

Inertia $\approx$ mass, distribution of mass, distribution of area

- Introduce the parallel-axis theorem.

- Be able to compute the moments of inertia of composite areas.

Applications

Modulus of Elasticity
Young's Modulus

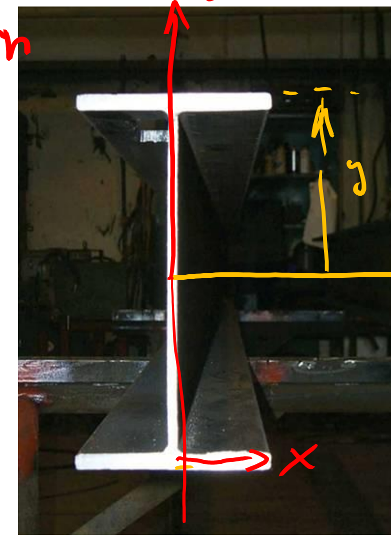
Y, E_m, $\boxed{E}$

These are aside notes from in-class discussion

Relates to:

moment of inertia

- x · shape orientation
- ☐ · mat'l property
- x · x Sect area



Many structural members like beams and columns have cross sectional shapes like an I, H, C, etc.

Why do they usually not have solid rectangular, square, or circular cross sectional areas?

What primary property of these members influences design decisions?

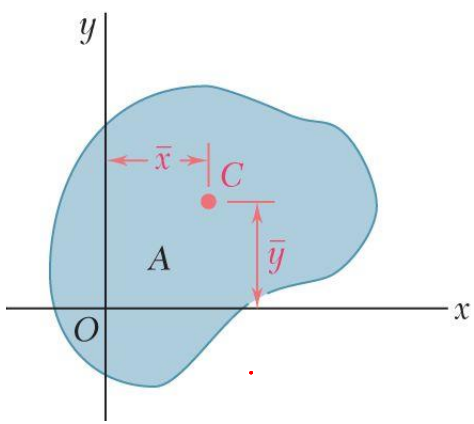
Applications

Many structural members are made of tubes rather than solid squares or rounds. **Why?**

This section of the book covers some parameters of the cross sectional area that influence the designer's selection.

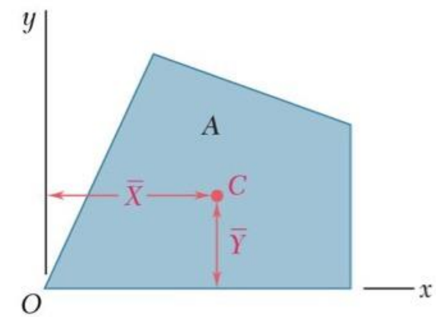
Recap: First moment of an area (centroid of an area)

- The first moment of the area A with respect to the x-axis is given by $Q_x = \int_A y\, dA$
- The first moment of the area A with respect to the y-axis is given by $Q_y = \int_A x\, dA$
- The centroid of the area A is defined as the point C of coordinates and , which satisfies the relation



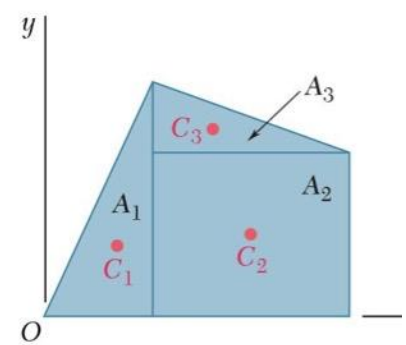
$$\int_A x\, dA = A\,\bar{x}$$

$$\int_A y\, dA = A\,\bar{y}$$



- In the case of a composite area, we divide the area A into parts

$$A_{total}\,\bar{X} = \sum_i A_i\,\bar{x}_i \qquad A_{total}\,\bar{Y} = \sum_i A_i\,\bar{y}_i$$



Terminology: the term **moment** in this module refers to the mathematical sense of different "measures" of an area or volume.

- The *zeroth* moment is the total mass./area
- The *first* moment (a single power of position) gave us the centroid.
- The *second* moment will allow us to describe the "width."
- An analogy that may help: in *probability* the first moment gives you the mean (the center of the distribution), and the second is the standard deviation (the width of the distribution).

Second moment of area

First moment of area
$Q_x = \int_A \tilde{y}\, dA$
$Q_y = \int_A \tilde{x}\, dA$

"Second moment of area" ≈ "Area moment of inertia"; note differences in names, but they both represent the same concept.

Moment of inertia is the property of a deformable body that determines the moment needed to obtain a desired curvature about an axis.
Moment of inertia depends on the shape of the body and may be different around different axes of rotation.

a.k.a: "Area moment of inertia"

- The moment of inertia of the area A with respect to the x-axis is given by

$$I_x = \int_A y^2\, dA$$

- The moment of inertia of the area A with respect to the y-axis is given by

$$I_y = \int_A x^2\, dA$$

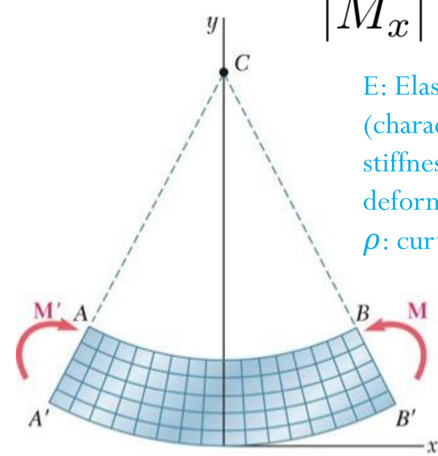
- The moment of inertia of the area A with respect to the origin *O* is given by (Polar moment of inertia)

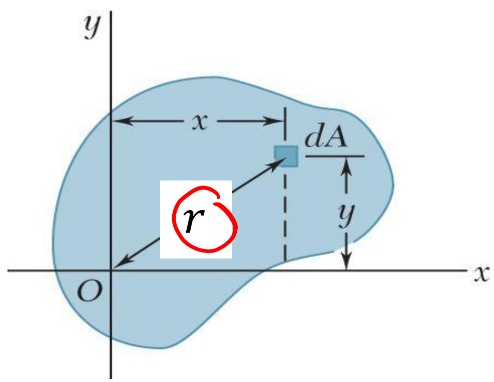
$$J_O = \int_A r^2\, dA = \int_A (x^2 + y^2)\, dA = I_y + I_x$$

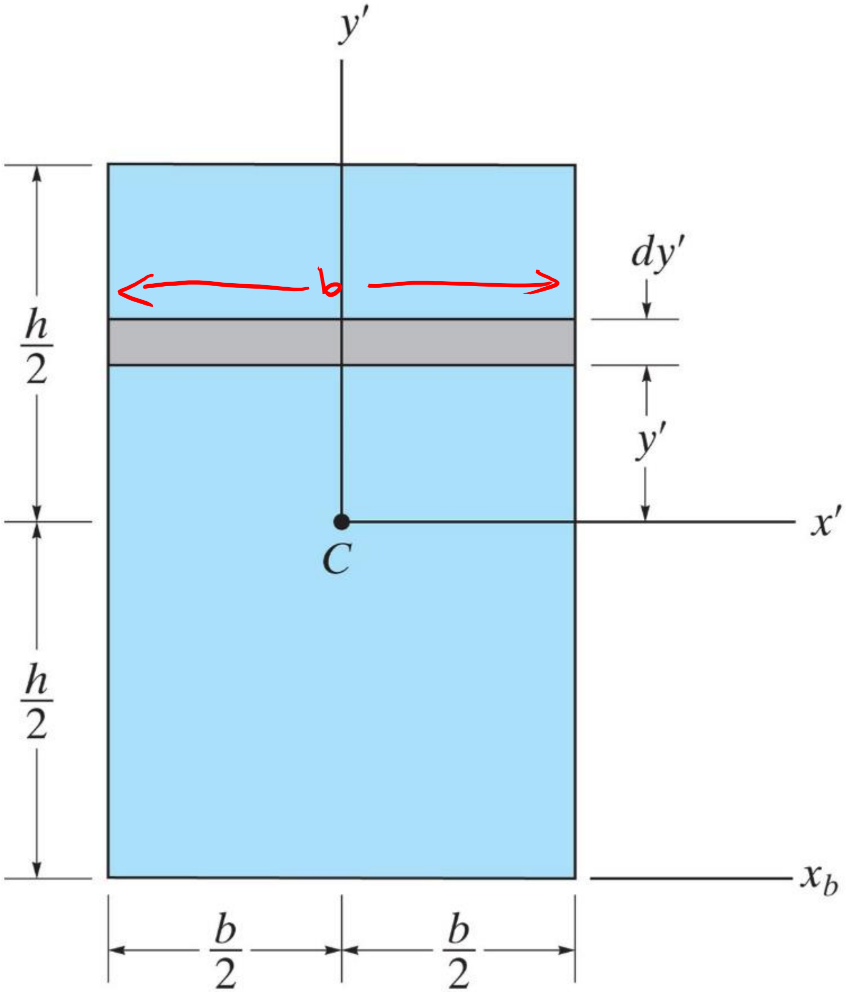
Moment-curvature relation:

$$|M_x| = \frac{E\, I_x}{\rho}$$

E: Elasticity modulus (characterizes stiffness of the deformable body)
ρ: curvature







Determine the moment of inertia for the rectangular area shown w.r.t. the centroidal axis x'.

$$I_{x'} = \int (y')^2 \, dA = \int (y')^2 (b\,dy)$$

$$= b \int_{-h/2}^{h/2} (y')^2 \, dy$$

$$= b \frac{(y')^3}{3} \Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{b}{3}\left(\left(\frac{h}{2}\right)^3 - \left(\frac{-h}{2}\right)^3\right)$$

$$\boxed{I_{x'} = \frac{1}{12} b h^3}$$

Parallel axis theorem

- Often, the **moment of inertia** of an area is known for an axis passing through the **centroid**; e.g., x' and y':
- The moments around other axes can be computed from the known $I_{x'}$ and $I_{y'}$:

$$I_x = \int_{\text{area}} (y' + d_y)^2 \, dA$$

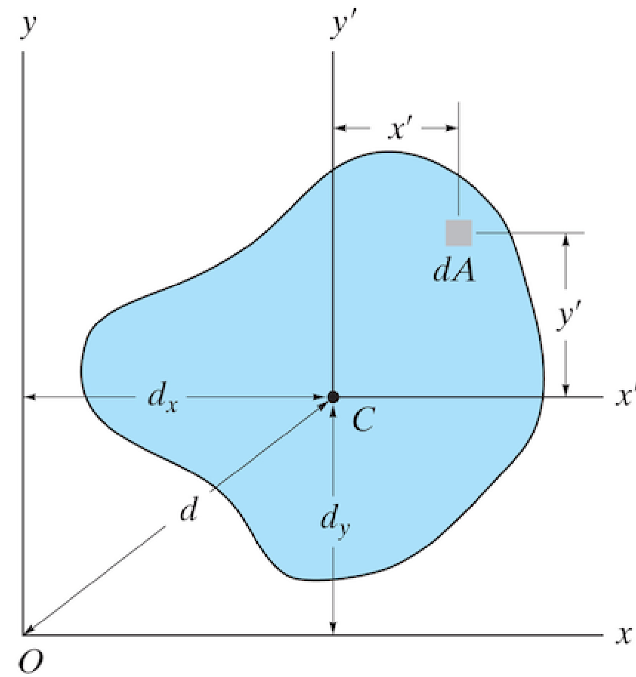
$$= \underbrace{\int_{\text{area}} (y')^2 \, dA}_{I_{x'}} + 2d_y \underbrace{\int_{\text{area}} y' \, dA}_{=0}$$

$$+ d_y^2 \int_{\text{area}} dA$$

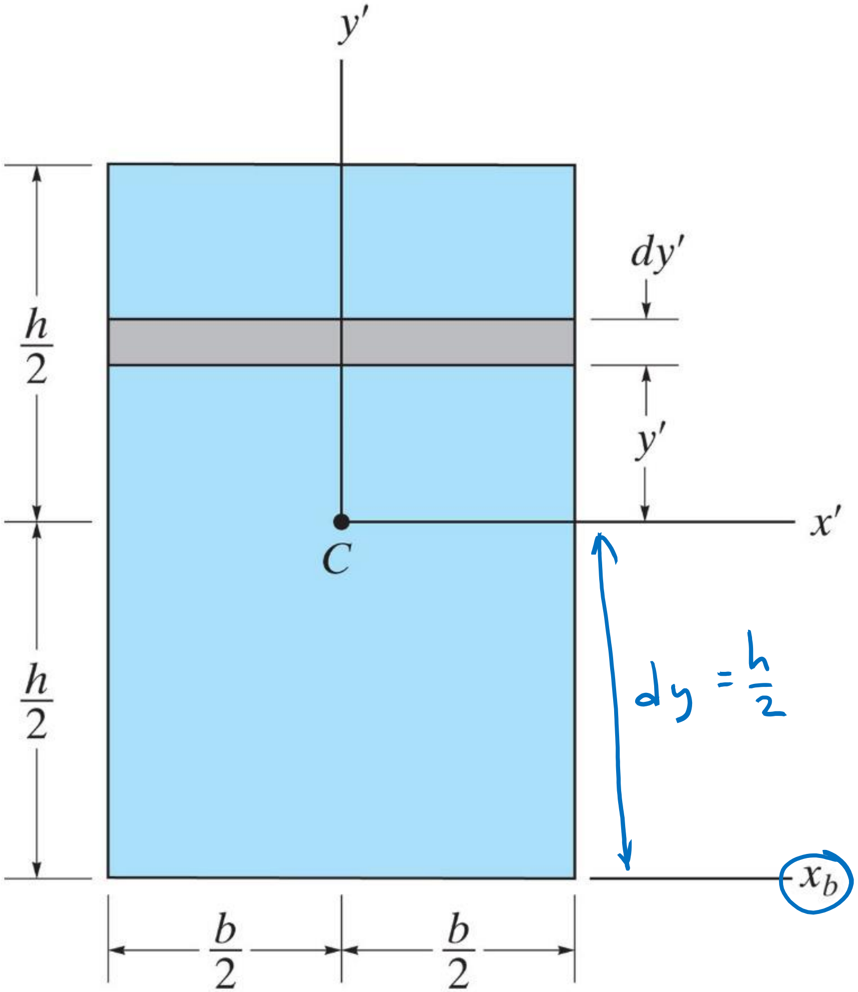
$$\boxed{I_x = I_{x'} + Ad_y^2}$$

$$\boxed{I_y = I_{y'} + Ad_x^2}$$

$$\boxed{J_O = J_C + A(d_x^2 + d_y^2) = J_C + Ad^2}$$



Note: the integral over y' gives zero *when done through the centroid axis*.



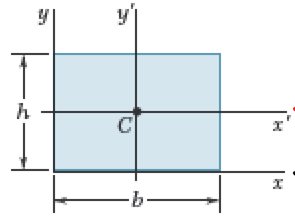
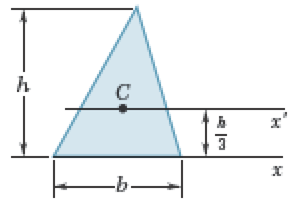
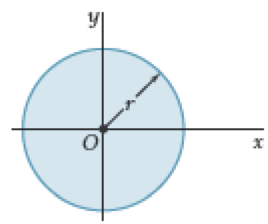
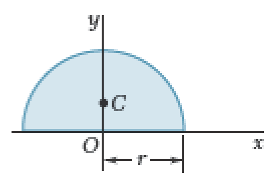
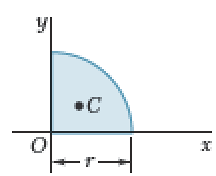
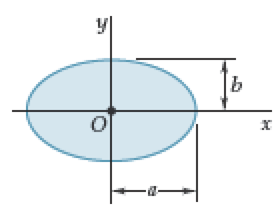
Determine the moment of inertia for the rectangular area shown w.r.t. the axis passing through the base of the rectangle x_b.

$$I_{x_b} = I_{x'} + A(d_y)^2$$

$$= \frac{1}{12}bh^3 + (bh)\left(\frac{h}{2}\right)^2$$

$$\boxed{I_{x_b} = \frac{1}{3}bh^3}$$

Area Moments of Inertia
for common shapes

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Eqn sheets will be provided in quizzes.

Need to see Quiz 0 for how to access attached documents at CBTF

Moment of inertia of composite

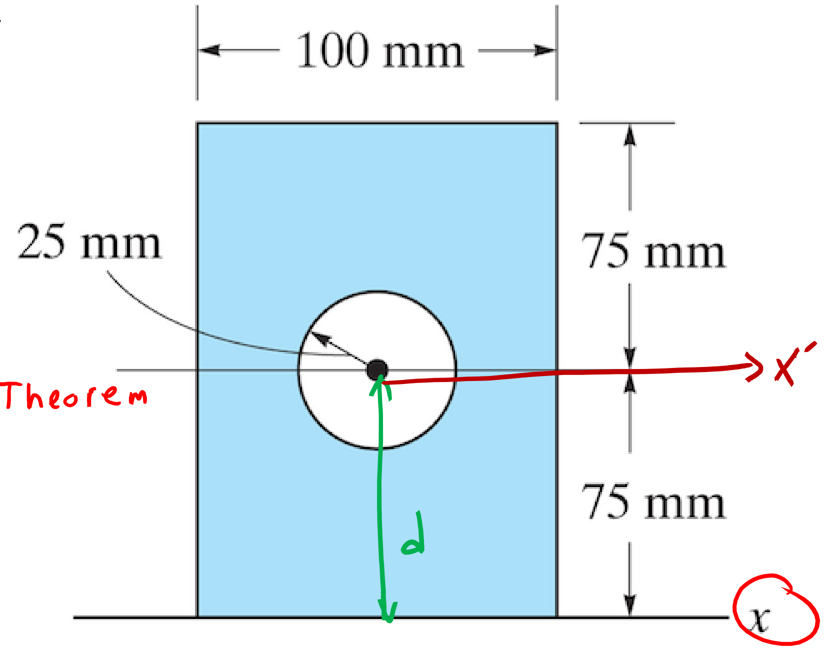
- If individual bodies making up a **composite** body have individual areas *A* and moments of inertia *I* computed through their centroids, then the **composite area** and **moment of inertia** is a sum of the individual component contributions.
- This requires the **parallel axis theorem**
- Remember:
  - The position of the centroid of each component **must** be defined with respect to the **same origin**.
  - It is allowed to consider **negative areas** in these expressions. Negative areas correspond to holes/missing area. **This is the one occasion to have negative moment of inertia**.

Determine the moment of inertia for the shaded area about the x-axis.

How many segments? 2 □ ○

About centroidal axis? No. ⇒ about x

⇒ Need to use Parallel Axis Theorem



$$I_x = I_{x\square} - I_{x\circ}$$

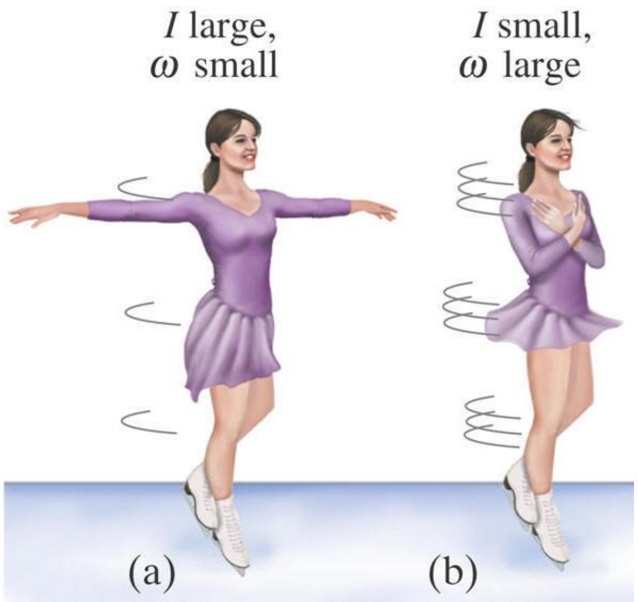
$$= \frac{1}{3} b h^3 - \left[I_{x'\circ} + A_\circ d^2 \right]$$

$$= \frac{1}{3} b h^3 - \left[\frac{1}{4} \pi r^4 + (\pi r^2) d^2 \right]$$

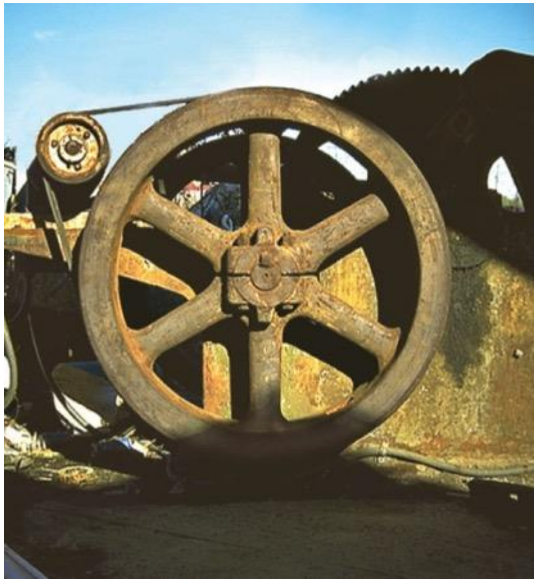
$$I_x = 101 (10^6) \text{ mm}^4$$

Mass Moment of Inertia

- **Mass moment of inertia** is the mass property of a rigid body that determines the torque *T* needed for a desired angular acceleration (α) about an axis of rotation.
- A larger mass moment of inertia around a given axis requires more torque to increase the rotation, or to stop the rotation, of a body about that axis
- Mass moment of inertia depends on the shape and density of the body and is different around different axes of rotation.



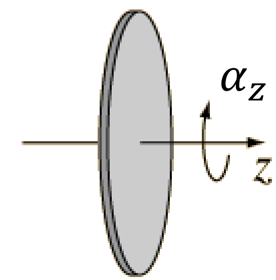
http://ffden-2.phys.uaf.edu/webproj/211_fall_2014/Ariel_Ellison/Ariel_Ellison/Angular.html



Mass Moment of Inertia

Mass Moment of Inertia

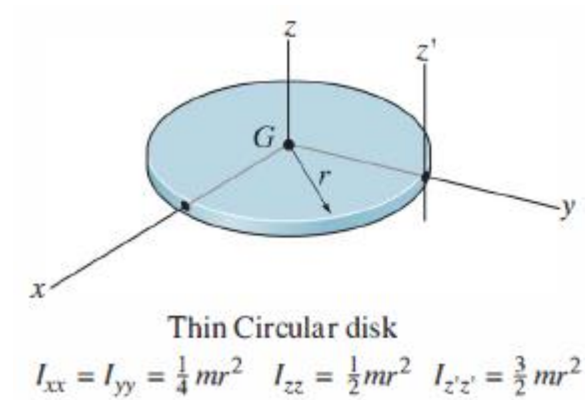
Torque-acceleration relation: $T = I\,\alpha$



where the mass moment of inertia is defined as

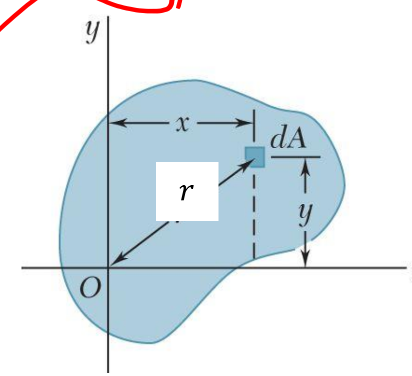
$$\boxed{I_{zz} = \int \rho\, r^2 dV}$$

$I_{zz} = \int r^2\, dm$, if constant ρ



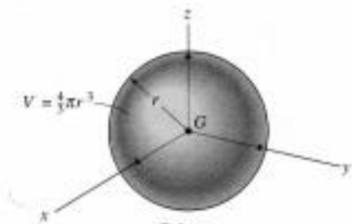
Mass moment of inertia for a disk:

$$\begin{aligned} I_{zz} &= \int \rho\, r^2\, dv = \int_0^t \int_0^{2\pi} \int_0^R \rho\, r^2\, (r\, dr\, d\theta\, dz) \\ &= \rho \int_0^t \int_0^{2\pi} \frac{r^4}{4} d\theta\, dz \\ &= \rho \int_0^t \frac{r^4}{2} \pi\, dz = \rho \frac{r^4}{2} \pi\, t = \frac{r^2}{2} \rho\, \pi\, r^2\, t = \frac{r^2}{2} \rho V = \frac{r^2}{2} M \end{aligned}$$



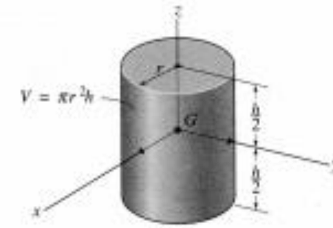
From inside back cover of Hibbler textbook

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



Sphere

$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$

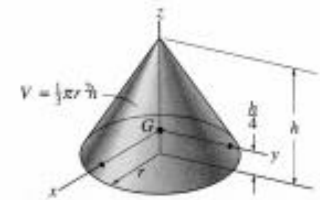


Cylinder

$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2)$ $I_{zz} = \frac{1}{2} mr^2$

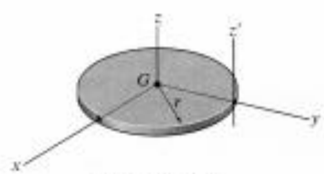
Hemisphere

$I_{xx} = I_{yy} = 0.259 mr^2$ $I_{zz} = \frac{2}{3} mr^2$



Cone

$I_{xx} = I_{yy} = \frac{3}{80} m(4r^2 + h^2)$ $I_{zz} = \frac{3}{10} mr^2$

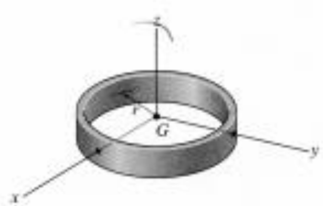


Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4} mr^2$ $I_{zz} = \frac{1}{2} mr^2$ $I_{z'z'} = \frac{3}{2} mr^2$

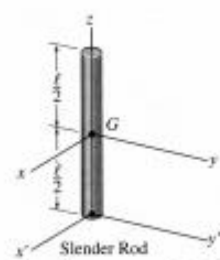
Thin plate

$I_{xx} = \frac{1}{12} mb^2$ $I_{yy} = \frac{1}{12} ma^2$ $I_{zz} = \frac{1}{12} m(a^2 + b^2)$



Thin ring

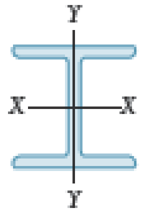
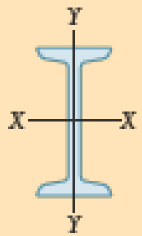
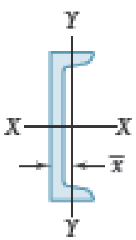
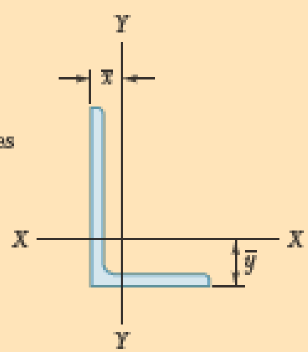
$I_{xx} = I_{yy} = \frac{1}{2} mr^2$ $I_{zz} = mr^2$



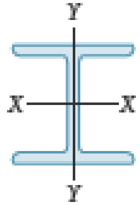
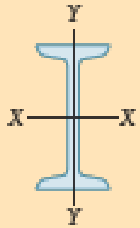
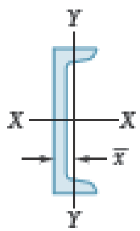
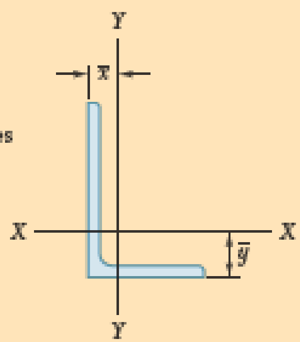
Slender Rod

$I_{xx} = I_{yy} = \frac{1}{12} m\ell^2$ $I_{x'x'} = I_{y'y'} = \frac{1}{3} m\ell^2$ $I_{z'z'} = 0$

English units (inches)

	Designation	Area in²	Depth in.	Width in.	Axis X-X $\bar{I}_x$, in⁴	Axis X-X $\bar{k}_x$, in.	Axis X-X $\bar{y}$, in.	Axis Y-Y $\bar{I}_y$, in⁴	Axis Y-Y $\bar{k}_y$, in.	Axis Y-Y $\bar{x}$, in.
W Shapes (Wide-Flange Shapes) 	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes) 	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.950	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels) 	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512
Angles 	L6 × 6 × 1‡	11.0			35.4	1.79	1.86	35.4	1.79	1.86
	L4 × 4 × $\frac{1}{2}$	3.75			5.52	1.21	1.18	5.52	1.21	1.18
	L3 × 3 × $\frac{1}{4}$	1.44			1.23	0.926	0.836	1.23	0.926	0.836
	L6 × 4 × $\frac{1}{2}$	4.75			17.3	1.91	1.98	6.22	1.14	0.981
	L5 × 3 × $\frac{1}{2}$	3.75			9.43	1.58	1.74	2.55	0.824	0.746
	L3 × 2 × $\frac{1}{4}$	1.19			1.09	0.953	0.980	0.390	0.569	0.487

Metric units (mm)

	Designation	Area mm²	Depth mm	Width mm	Axis X-X: $\bar{I}_x$ 10^6 mm⁴	Axis X-X: $\bar{k}_x$ mm	Axis X-X: $\bar{y}$ mm	Axis Y-Y: $\bar{I}_y$ 10^6 mm⁴	Axis Y-Y: $\bar{k}_y$ mm	Axis Y-Y: $\bar{x}$ mm
W Shapes (Wide-Flange Shapes) 	W460 × 113†	14400	462	279	554	196		63.3	66.3	
	W410 × 85	10800	417	181	316	171		17.9	40.6	
	W360 × 57.8	7230	358	172	160	149		11.1	39.4	
	W200 × 46.1	5880	203	203	45.8	88.1		15.4	51.3	
S Shapes (American Standard Shapes) 	S460 × 81.4†	10300	457	152	333	180		8.62	29.0	
	S310 × 47.3	6010	305	127	90.3	123		3.88	25.4	
	S250 × 37.8	4810	254	118	51.2	103		2.80	24.1	
	S150 × 18.6	2360	152	84.6	9.16	62.2		0.749	17.8	
C Shapes (American Standard Channels) 	C310 × 30.8†	3920	305	74.7	53.7	117		1.61	20.2	17.7
	C250 × 22.8	2890	254	66.0	28.0	98.3		0.945	18.1	16.1
	C200 × 17.1	2170	203	57.4	13.5	79.0		0.545	15.8	14.5
	C150 × 12.2	1540	152	48.8	5.45	59.4		0.286	13.6	13.0
Angles 	L152 × 152 × 25.4‡	7100			14.7	45.5	47.2	14.7	45.5	47.2
	L102 × 102 × 12.7	2420			2.30	30.7	30.0	2.30	30.7	30.0
	L76 × 76 × 6.4	929			0.512	23.5	21.2	0.512	23.5	21.2
	L152 × 102 × 12.7	3060			7.20	48.5	50.3	2.59	29.0	24.9
	L127 × 76 × 12.7	2420			3.93	40.1	44.2	1.06	20.9	18.9
	L76 × 51 × 6.4	768			0.454	24.2	24.9	0.162	14.5	12.4