Statics - TAM 211

Lecture 32

December 12, 2018

Chap 10.1, 10.2, 10.4, 10.8

Announcements

- Check ALL of your grades on Blackboard! Report issues
- Prof. H-W office hours
  - Monday 3-5pm (Room C315 ZJUI Building)
  - Wednesday 7-8pm (Residential College Lobby)
- Upcoming deadlines:
  - Friday (12/14)
    - Written Assignment 12
  - Friday (12/14) all in Teaching Building A418-420
    - **9:00 am:** Quiz 6, On paper. Chapter 9 (CoG thru Fluid Pressure)
    - 10:00 am: Discussion section for ALL students
    - No lecture to accommodate quiz 6 testing time
  - Tuesday (12/18)
    - HW 13

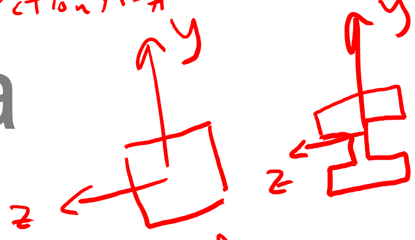
Chapter 10: Moments of Inertia

Goals and Objectives

- Understand the term "moment" as used in this chapter
- Determine and know the differences between
  - First/second moment of area
  - Moment of inertia for an area
  - Polar moment of inertia
  - Mass moment of inertia
- Introduce the parallel-axis theorem.
- Be able to compute the moments of inertia of composite areas.

Recap: Area moment of inertia (Second moment of area)

Section A-A

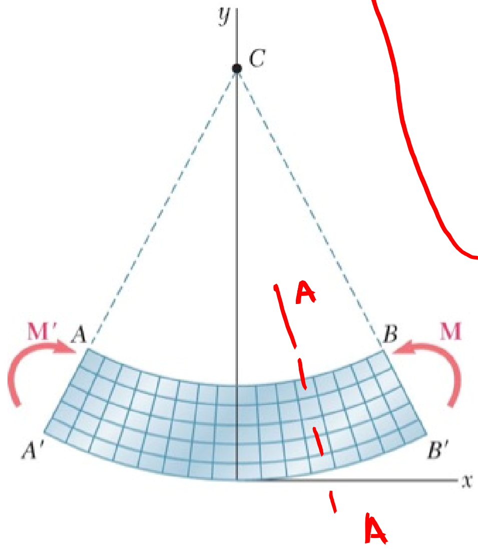


$I_z = \int_A y^2 dA$

- The moment of inertia of the area A with respect to the x-axis is given by
$$I_x = \int_A y^2\, dA$$
- The moment of inertia of the area A with respect to the y-axis is given by
$$I_y = \int_A x^2\, dA$$
- The moment of inertia of the area A with respect to the origin O is given by (Polar moment of inertia)
$$J_O = \int_A r^2\, dA = \int_A (x^2 + y^2)\, dA = I_y + I_x$$

Moment-curvature relation:

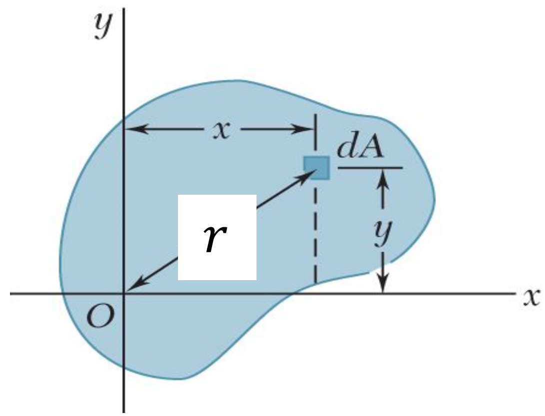
$$|M_z| = \frac{EI_z}{\rho}$$



I_z: Moment of inertia of cross-sectional area

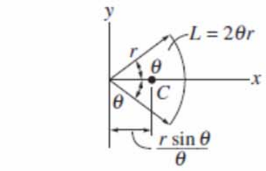
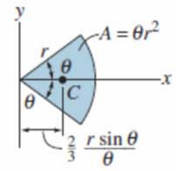
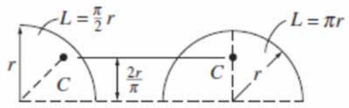
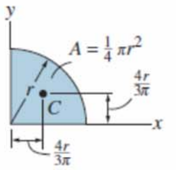
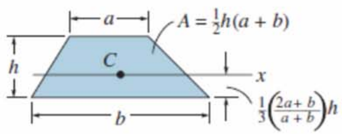
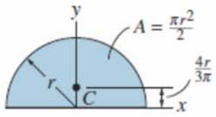
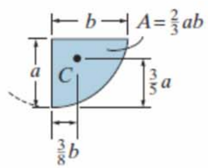
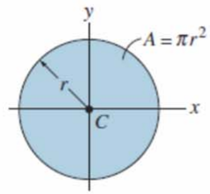
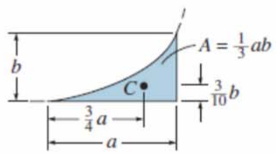
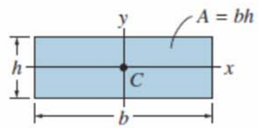
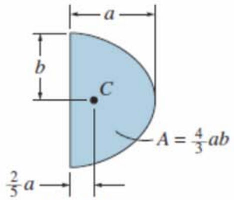
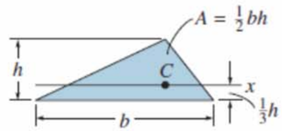
E: Elasticity modulus (characterizes stiffness of the deformable body)

ρ: curvature



From inside back cover of Hibbler textbook

Geometric Properties of Line and Area Elements

Centroid Location	Centroid Location	Area Moment of Inertia
 Circular arc segment	 Circular sector area	$I_x = \frac{1}{4} r^4 (\theta - \frac{1}{2}\sin 2\theta)$ $I_y = \frac{1}{4} r^4 (\theta + \frac{1}{2}\sin 2\theta)$
 Quarter and semicircle arcs	 Quarter circle area	$I_x = \frac{1}{16}\pi r^4$ $I_y = \frac{1}{16}\pi r^4$
 Trapezoidal area	 Semicircular area	$I_x = \frac{1}{8}\pi r^4$ $I_y = \frac{1}{8}\pi r^4$
 Semiparabolic area	 Circular area	$I_x = \frac{1}{4}\pi r^4$ $I_y = \frac{1}{4}\pi r^4$
 Exparabolic area	 Rectangular area	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}hb^3$
 Parabolic area	 Triangular area	$I_x = \frac{1}{36}bh^3$

$[mm^4]$, $[m^4]$

I_{area} has units of $[\text{length}]^4$

cf $F = ma$ for linear or translational motion

For rotary or rotational motion

Mass Moment of Inertia

$T = I\alpha$

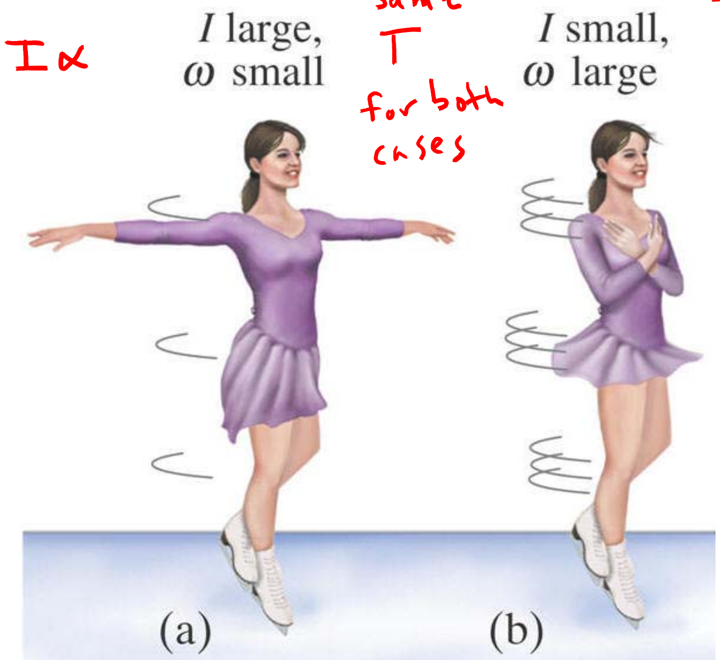
↑ units of $[\text{mass}][\text{length}]^2$

Mass Moment of Inertia

- **Mass moment of inertia** is the mass property of a rigid body that determines the torque T needed for a desired angular acceleration (α) about an axis of rotation.
- A larger mass moment of inertia around a given axis requires more torque to increase the rotation, or to stop the rotation, of a body about that axis
- Mass moment of inertia depends on the shape and density of the body and is different around different axes of rotation.

I_{mass} is used in dynamics TAM 212

$T_1 = I\alpha$



Same T for both cases

$T = T_1$

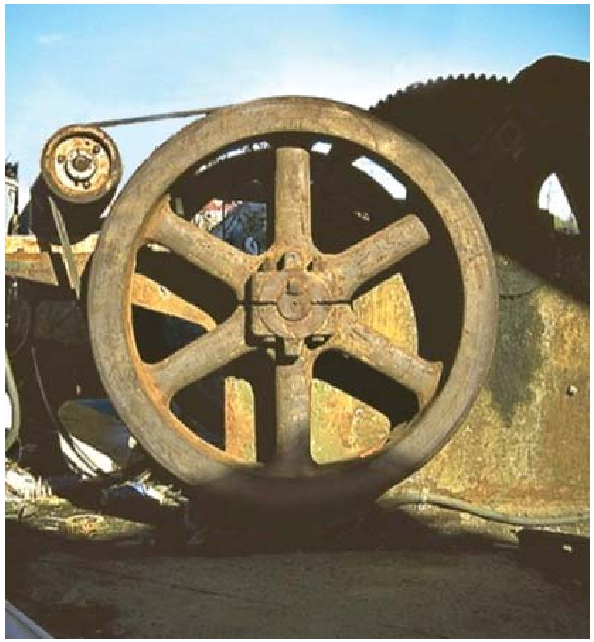
If $I \downarrow$

$\Rightarrow \alpha \uparrow$

$\therefore \omega \uparrow$

so skater spins faster with tighter (smaller) I_{mass}

http://ffden-2.phys.uaf.edu/webproj/211_fall_2014/Ariel_Ellison/Ariel_Ellison/Angular.html



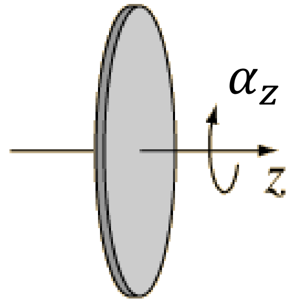
Mass Moment of Inertia

Torque-acceleration relation: $T = I\,\alpha$

where the mass moment of inertia is defined as

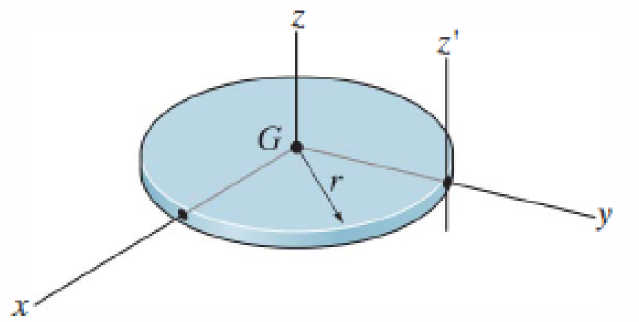
$$I_{zz} = \int \rho\, r^2 dV$$

If $\rho =$ constant, $\boxed{I_{zz} = \int r^2 dm}$



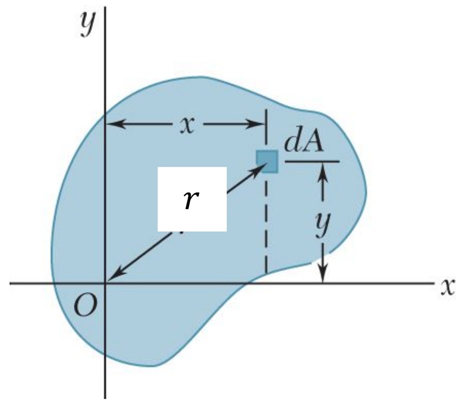
Mass moment of inertia for a disk:

$$\begin{aligned}
I_{zz} &= \int \rho\, r^2\, dv = \int_0^t \int_0^{2\pi} \int_0^R \rho\, r^2 \,(r\, dr\, d\theta\, dz) \\
&= \rho \int_0^t \int_0^{2\pi} \frac{r^4}{4} d\theta\, dz \\
&= \rho \int_0^t \frac{r^4}{2} \pi\, dz = \rho \frac{r^4}{2} \pi\, t = \frac{r^2}{2} \rho\, \pi\, r^2\, t = \frac{r^2}{2} \rho\, V = \frac{r^2}{2} M
\end{aligned}$$



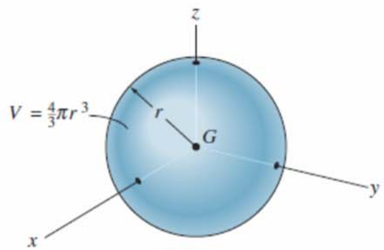
Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \quad I_{zz} = \frac{1}{2} mr^2 \quad I_{z'z'} = \frac{3}{2} mr^2$



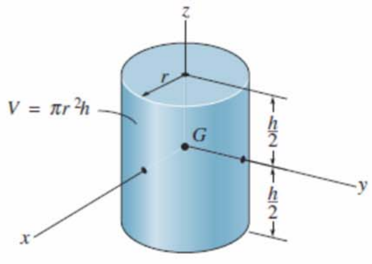
From inside back cover of Hibbler textbook

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



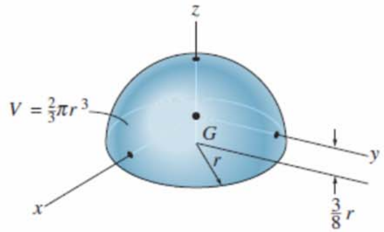
Sphere

$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$



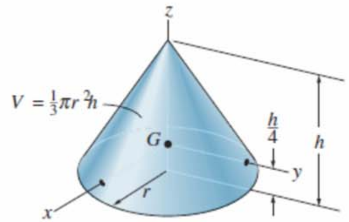
Cylinder

$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2) \quad I_{zz} = \frac{1}{2} mr^2$



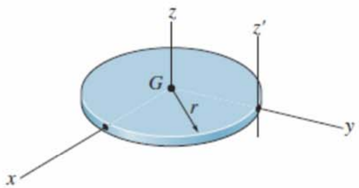
Hemisphere

$I_{xx} = I_{yy} = 0.259mr^2 \quad I_{zz} = \frac{2}{5}mr^2$



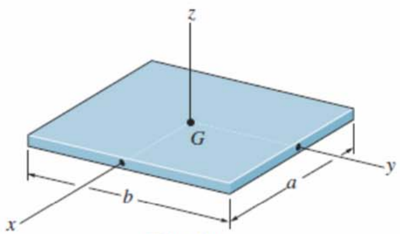
Cone

$I_{xx} = I_{yy} = \frac{3}{80} m (4r^2 + h^2) \quad I_{zz} = \frac{3}{10}mr^2$



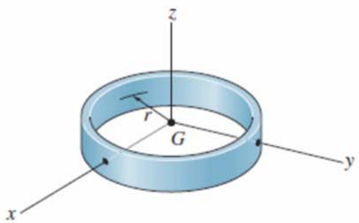
Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2} mr^2$



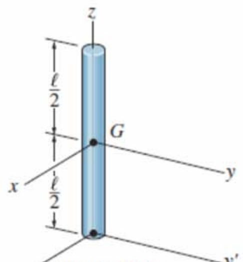
Thin plate

$I_{xx} = \frac{1}{12} mb^2 \quad I_{yy} = \frac{1}{12} ma^2 \quad I_{zz} = \frac{1}{12} m(a^2 + b^2)$



Thin ring

$I_{xx} = I_{yy} = \frac{1}{2} mr^2 \quad I_{zz} = mr^2$



Slender Rod

$I_{xx} = I_{yy} = \frac{1}{12} m\ell^2 \quad I_{x'x'} = I_{y'y'} = \frac{1}{3} m\ell^2 \quad I_{z'z'} = 0$

Recap: Parallel axis theorem

- Often, the **moment of inertia** of an area is known for an axis passing through the **centroid**; e.g., x' and y':
- The moments around other axes can be computed from the known $I_{x'}$ and $I_{y'}$:

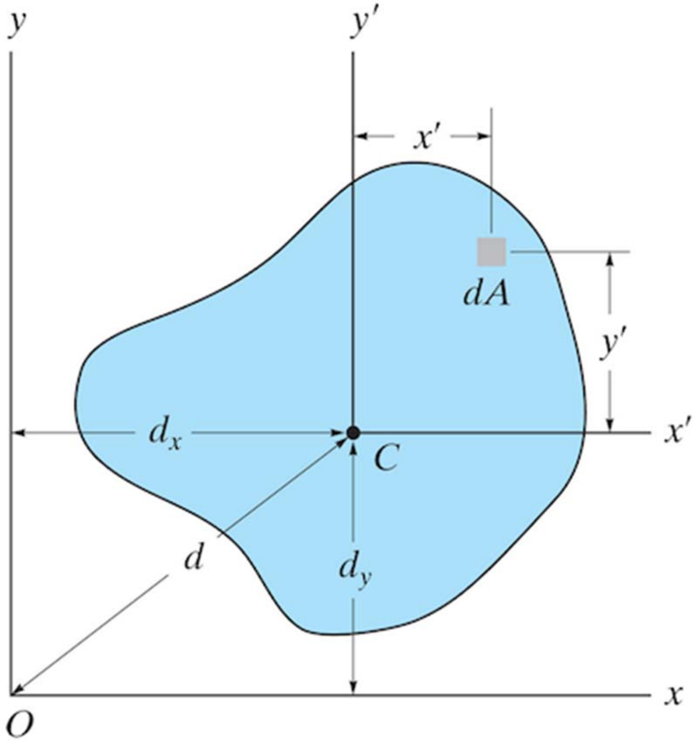
$$I_x = I_{x'} + Ad_y^2$$

$$I_y = I_{y'} + Ad_x^2$$

$$J_O = J_C + A(d_x^2 + d_y^2) = J_C + Ad^2$$

Moments of inertia relative to axis through CENTROID

Key Point!



Note: the integral over y' gives zero *when done through the centroid axis*.

For the uniform rectangular plate of area $A = l\,w$

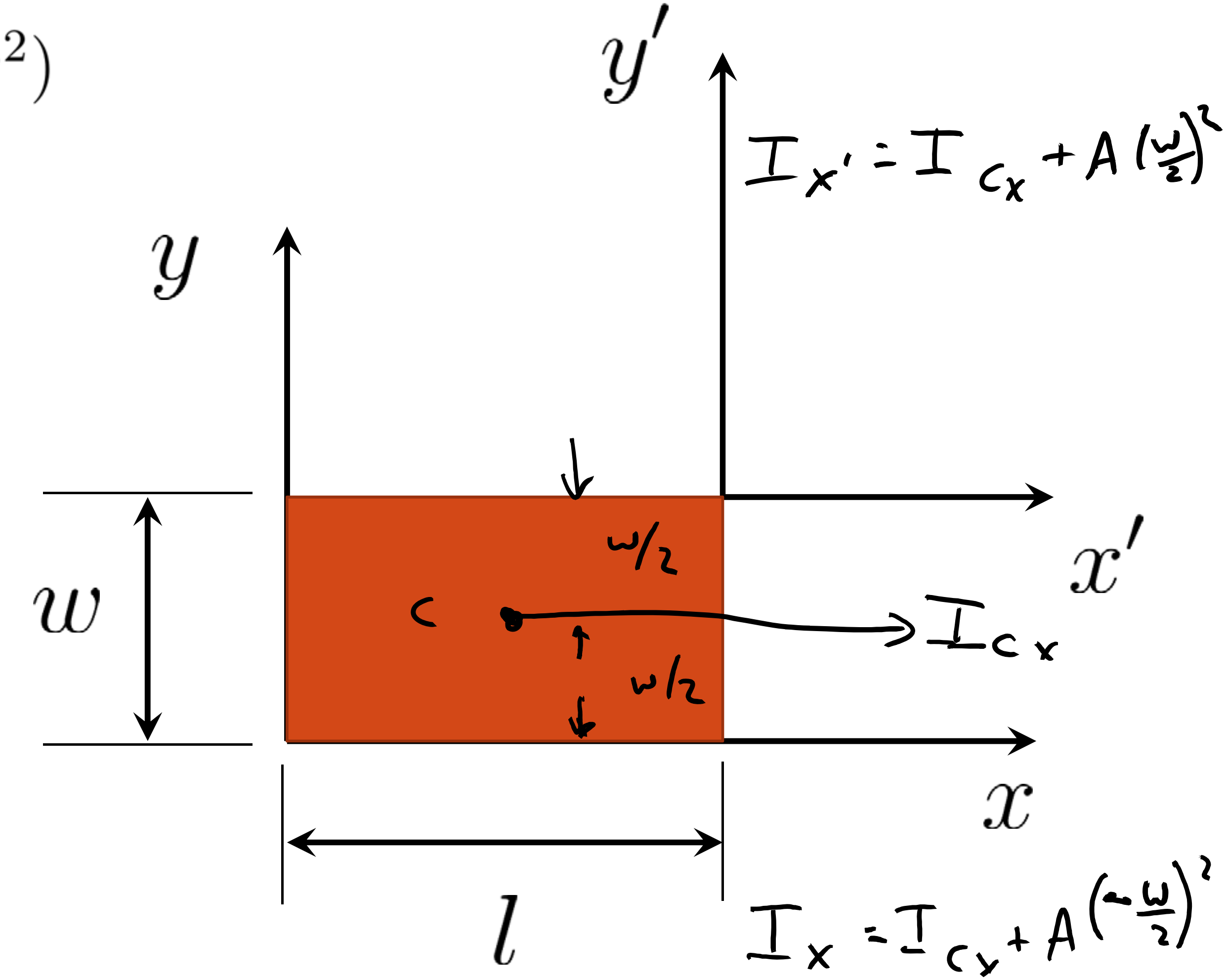
A) $I_{x'} = I_x + A\,l^2$

B) $I_{x'} = I_x + A\,w^2$

C) $I_{x'} = I_x + A\,(l^2 + w^2)$

D) $I_{x'} = I_x$

E) None of the above



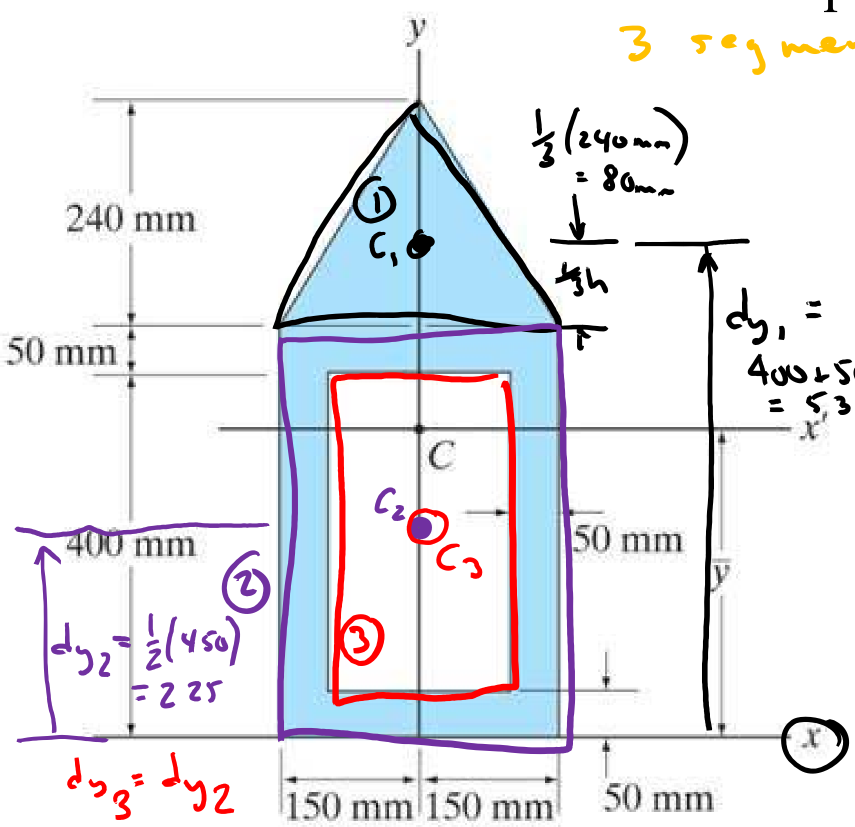
Recap: Moment of inertia of composite

- If individual bodies making up a **composite** body have individual areas *A* and moments of inertia *I* computed through a common axis, then the **composite area** and **moment of inertia** is a sum of the individual component contributions about the axis

FIND I_x for given shape

This requires the **parallel axis theorem**:

3 segments



$\triangle + \square - \square = I_x$

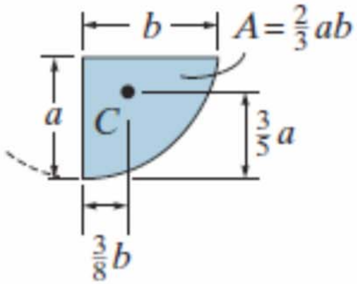
$I_{x_1} + I_{x_2} - I_{x_3} = I_x$

$$I_{x_1} = \bar{I}_{x'_{Tri}} + A_T (d_{y_1})^2 = \frac{1}{36}(300)(240)^3 + \frac{1}{2}(300)(240)(530)^2$$

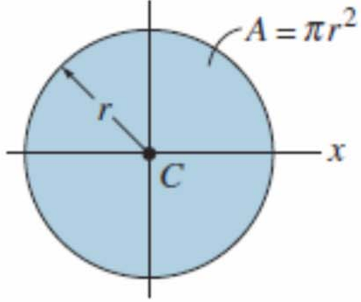
$$I_{x_2} = \bar{I}_{x'_2} + A_2 (d_{y_2})^2 = \frac{1}{12}(300)(450)^3 + (300)(450)(225)^2$$

$$I_{x_3} = \bar{I}_{x'_3} + A_3 (d_{y_3})^2 = \frac{1}{12}(200)(350)^3 + (200)(350)(225)^2$$

$$I_x = I_{x_1} + I_{x_2} - I_{x_3}$$



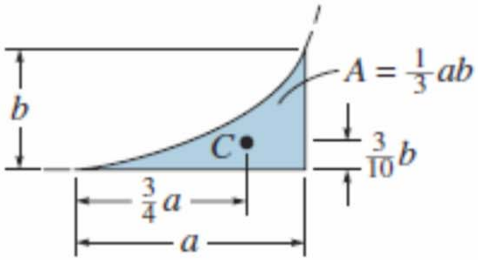
Semiparabolic area



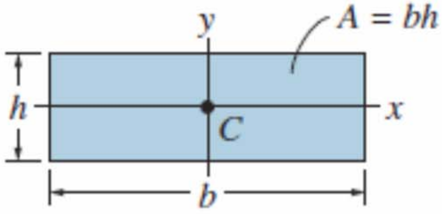
Circular area

$$I_x = \frac{1}{4}\pi r^4$$

$$I_y = \frac{1}{4}\pi r^4$$



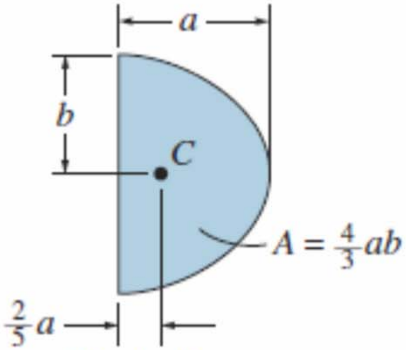
Exparabolic area



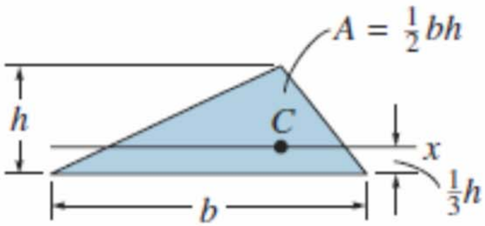
Rectangular area

$$I_x = \frac{1}{12}bh^3$$

$$I_y = \frac{1}{12}hb^3$$



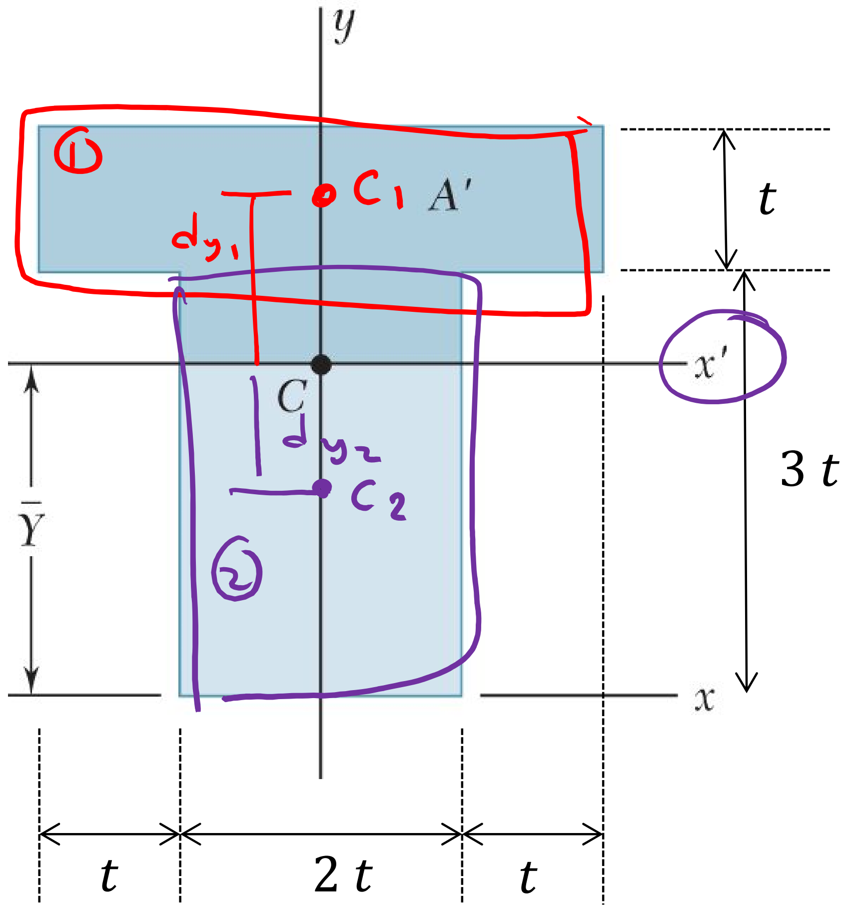
Parabolic area



Triangular area

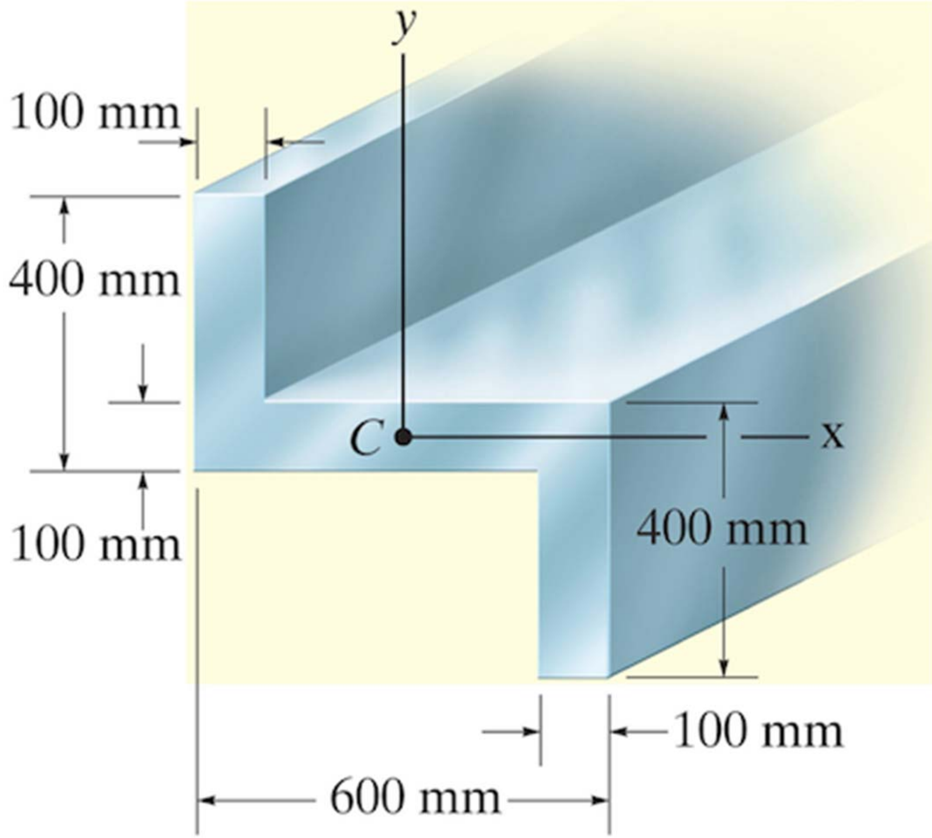
$$I_x = \frac{1}{36}bh^3$$

Find the moment of inertia of the shape about its centroid:



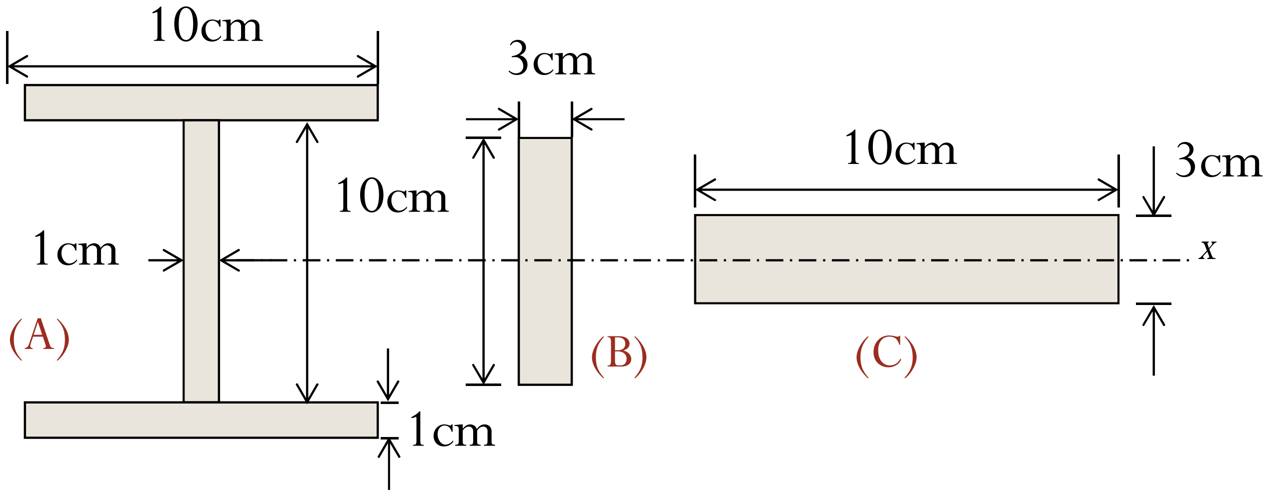
$$I_{x'} = I_{x'_1} + I_{x'_2}$$

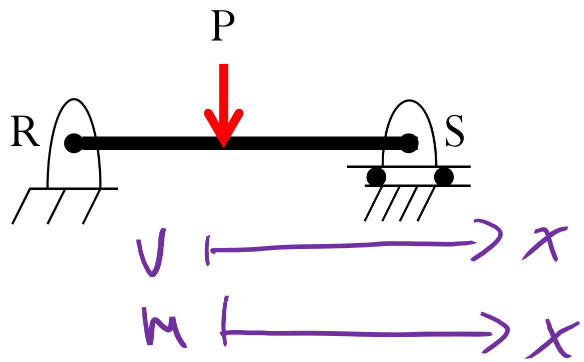
Determine the moment of inertia for the cross-sectional area about the x and y centroidal axes.



see text
for solution

Example 10.5 in text





Consider three different possible cross sectional shapes and areas for the beam RS. For the given vertical loading P on the beam, which shape will develop less internal stress and deflection?

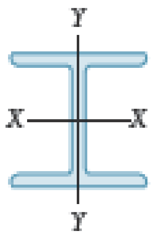
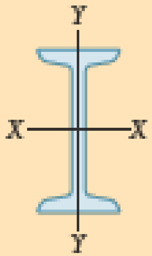
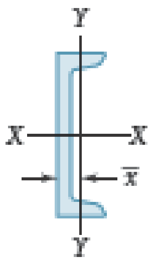
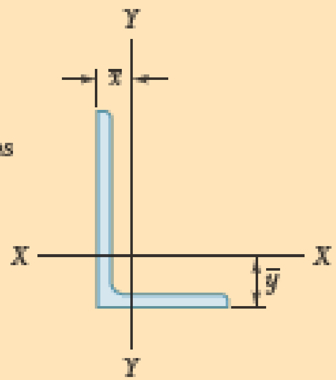
Consider this problem on your own.

1. Can you draw V & M diagrams for the beam RS?
2. Recall $M = \frac{EI}{\rho}$ (see slide #5)
3. Compute I for each cross sectional shape

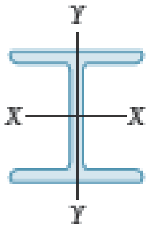
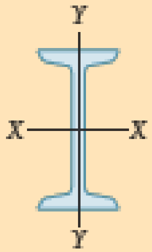
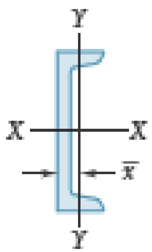
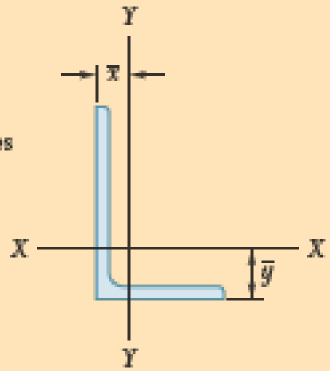
This question is similar to questions in TAM 251.

English units (inches)

Tables of Cross-sections of Common Structural Members

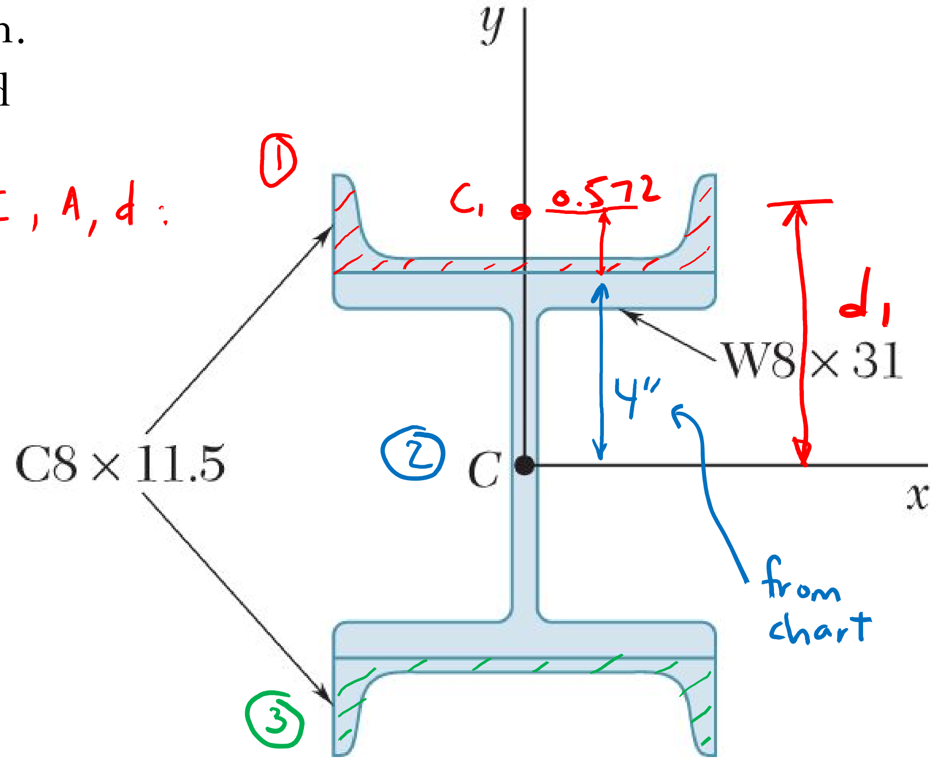
	Designation	Area in²	Depth in.	Width in.	Axis X-X $\bar{I}_x$, in⁴	Axis X-X $\bar{k}_x$, in.	Axis X-X $\bar{y}$, in.	Axis Y-Y $\bar{I}_y$, in⁴	Axis Y-Y $\bar{k}_y$, in.	Axis Y-Y $\bar{x}$, in.
W Shapes (Wide-Flange Shapes) 	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes) 	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.980	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels) 	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512
Angles 	L6 × 6 × 1‡	11.0			35.4	1.79	1.86	35.4	1.79	1.86
	L4 × 4 × $\frac{1}{2}$	3.75			5.52	1.21	1.18	5.52	1.21	1.18
	L3 × 3 × $\frac{1}{4}$	1.44			1.23	0.926	0.836	1.23	0.926	0.836
	L6 × 4 × $\frac{1}{2}$	4.75			17.3	1.91	1.98	6.22	1.14	0.981
	L5 × 3 × $\frac{1}{2}$	3.75			9.43	1.58	1.74	2.55	0.824	0.746
	L3 × 2 × $\frac{1}{4}$	1.19			1.09	0.953	0.980	0.390	0.569	0.487

Metric units (mm)

	Designation	Area mm²	Depth mm	Width mm	Axis X-X $\bar{I}_x$ 10^6 mm⁴	Axis X-X $\bar{k}_x$ mm	Axis X-X $\bar{y}$ mm	Axis Y-Y $\bar{I}_y$ 10^6 mm⁴	Axis Y-Y $\bar{k}_y$ mm	Axis Y-Y $\bar{x}$ mm
W Shapes (Wide-Flange Shapes) 	W460 × 113†	14400	462	279	554	196		63.3	66.3	
	W410 × 85	10800	417	181	316	171		17.9	40.6	
	W360 × 57.8	7230	358	172	160	149		11.1	39.4	
	W200 × 46.1	5880	203	203	45.8	88.1		15.4	51.3	
S Shapes (American Standard Shapes) 	S460 × 81.4†	10300	457	152	333	180		8.62	29.0	
	S310 × 47.3	6010	305	127	90.3	123		3.88	25.4	
	S250 × 37.8	4810	254	118	51.2	103		2.80	24.1	
	S150 × 18.6	2360	152	84.6	9.16	62.2		0.749	17.8	
C Shapes (American Standard Channels) 	C310 × 30.8†	3920	305	74.7	53.7	117		1.61	20.2	17.7
	C250 × 22.8	2890	254	66.0	28.0	98.3		0.945	18.1	16.1
	C200 × 17.1	2170	203	57.4	13.5	79.0		0.545	15.8	14.5
	C150 × 12.2	1540	152	48.8	5.45	59.4		0.286	13.6	13.0
Angles 	L152 × 152 × 25.4‡	7100			14.7	45.5	47.2	14.7	45.5	47.2
	L102 × 102 × 12.7	2420			2.30	30.7	30.0	2.30	30.7	30.0
	L76 × 76 × 6.4	929			0.512	23.5	21.2	0.512	23.5	21.2
	L152 × 102 × 12.7	3060			7.20	48.5	50.3	2.59	29.0	24.9
	L127 × 76 × 12.7	2420			3.93	40.1	44.2	1.06	20.9	18.9
	L76 × 51 × 6.4	768			0.454	24.2	24.9	0.162	14.5	12.4

Two channels are welded to a rolled W section as shown. Determine the area moments of inertia of the combined section with respect to the centroidal x and y axes.

Use chart and part designations to determine I, A, d:



$$I_x = I_{x_1} + I_{x_2} + I_{x_3}$$

$$I_{x_1} = \left[1.31 + A(d_1)^2\right]$$

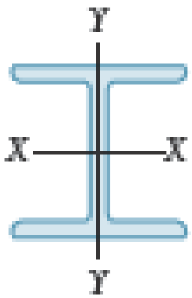
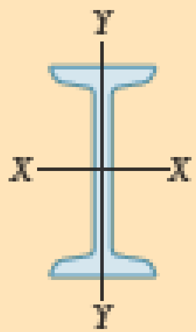
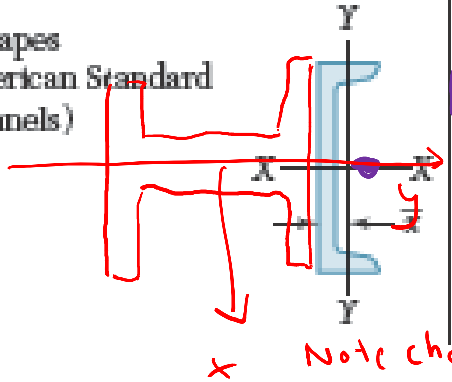
$$= \left[1.31 + (3.37)(4 + 0.572)^2\right]$$

$$= 71.8 \text{ in}^4$$

$$I_x = 2(71.8) + 110 \text{ in}^4 = 254 \text{ in}^4$$

$$I_y = 102.1 \text{ in}^4$$

English units (inches)

	Designation	Area in²	Depth in.	Width in.	Axis X-X: $\bar{I}_x$, in⁴	Axis X-X: $\bar{k}_x$, in.	Axis X-X: $\bar{y}$, in.	Axis Y-Y: $\bar{I}_y$, in⁴	Axis Y-Y: $\bar{k}_y$, in.	Axis Y-Y: $\bar{x}$, in.
W Shapes (Wide-Flange Shapes) 	W18 × 76†	22.3	18.2	11.0	1330	7.73		152	2.61	
	W16 × 57	16.8	16.4	7.12	758	6.72		43.1	1.60	
	W14 × 38	11.2	14.1	6.77	385	5.87		26.7	1.55	
	W8 × 31	9.12	8.00	8.00	110	3.47		37.1	2.02	
S Shapes (American Standard Shapes) 	S18 × 54.7†	16.0	18.0	6.00	801	7.07		20.7	1.14	
	S12 × 31.8	9.31	12.0	5.00	217	4.83		9.33	1.00	
	S10 × 25.4	7.45	10.0	4.66	123	4.07		6.73	0.950	
	S6 × 12.5	3.66	6.00	3.33	22.0	2.45		1.80	0.702	
C Shapes (American Standard Channels) 	C12 × 20.7†	6.08	12.0	2.94	129	4.61		3.86	0.797	0.698
	C10 × 15.3	4.48	10.0	2.60	67.3	3.87		2.27	0.711	0.634
	C8 × 11.5	3.37	8.00	2.26	32.5	3.11		1.31	0.623	0.572
	C6 × 8.2	2.39	6.00	1.92	13.1	2.34		0.687	0.536	0.512

I_y

I_x

0.572

Note change of axis orientation chart vs. problem