Statics - TAM 211

Lecture 33

December 17, 2018

Announcements

- Check ALL of your grades on Blackboard! Report issues
- Prof. H-W office hours
  - Monday 3-5pm – NO OFFICE HOURS THIS WEEK
  - Wednesday 7-8pm (Residential College Lobby)
- Upcoming deadlines:
  - Tuesday (12/18)
    - HW 13
  - Friday (12/21)
    - Written Assignment 13
  - Tuesday (12/25)
    - HW 14
  - Friday (12/28)
    - Written Assignment 14
- Final Exam – computer based
  - Wednesday January 9, 9:00-12:00
  - Instructional Lab Building: D211 (ME students), D331 (CEE students)

Chapter 10: Moments of Inertia

cf $F = ma$ for linear or translational motion

For rotary or rotational motion $T = I\alpha$

↑ units of [mass][length]^2 Mass Moment of Inertia

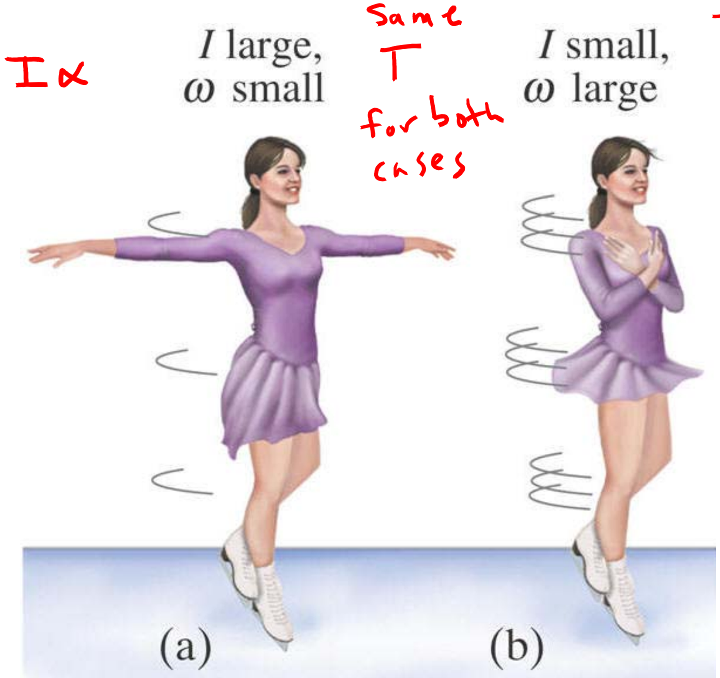
Recap: Mass Moment of Inertia

- **Mass moment of inertia** is the mass property of a rigid body that determines the torque T needed for a desired angular acceleration (α) about an axis of rotation.
- A larger mass moment of inertia around a given axis requires more torque to increase the rotation, or to stop the rotation, of a body about that axis
- Mass moment of inertia depends on the shape and density of the body and is different around different axes of rotation.

I_{mass} is used in dynamics TAM 212

$T_1 = I\alpha$

Same T for both cases



$T = T_1$

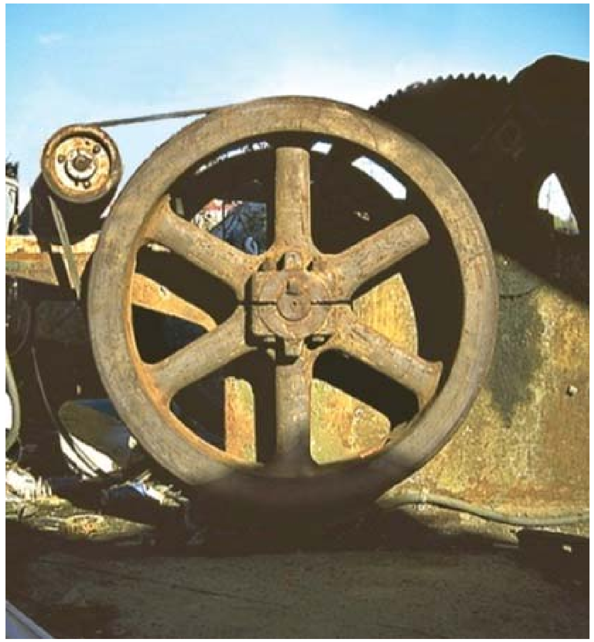
If $I \downarrow$

$\Rightarrow \alpha \uparrow$

$\therefore \omega \uparrow$

so skater spins faster with tighter (smaller) I_{mass}

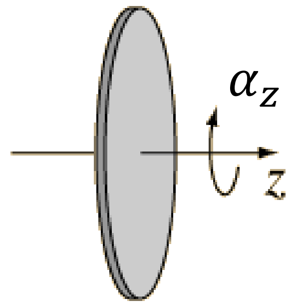
http://ffden-2.phys.uaf.edu/webproj/211_fall_2014/Ariel_Ellison/Ariel_Ellison/Angular.html



Recap: Mass Moment of Inertia

Mass Moment of Inertia

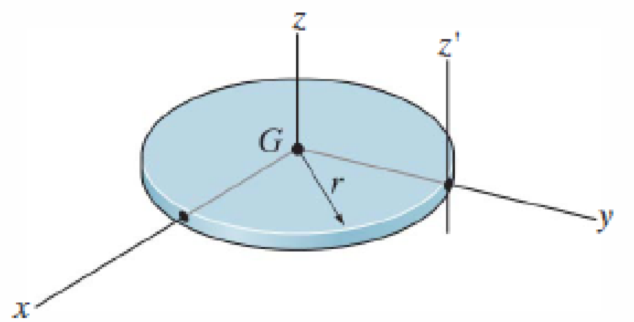
Torque-acceleration relation: $T = I\,\alpha$



where the mass moment of inertia is defined as

$$I_{zz} = \int \rho\, r^2 dV$$

$I_{zz} = \int r^2\, dm$, if constant ρ

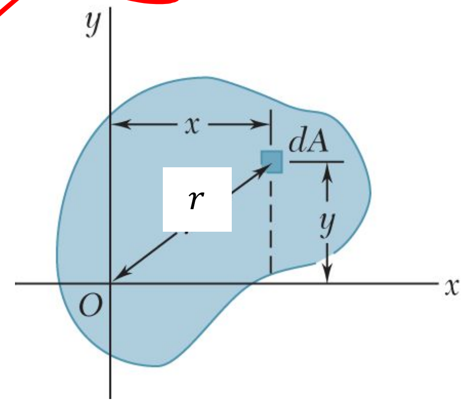


Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \quad I_{zz} = \frac{1}{2} mr^2 \quad I_{z'z'} = \frac{3}{2} mr^2$

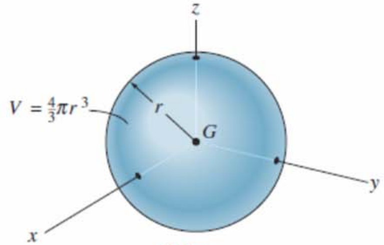
Mass moment of inertia for a disk:

$$\begin{aligned} I_{zz} &= \int \rho\, r^2\, dv = \int_0^t \int_0^{2\pi} \int_0^R \rho\, r^2\, (r\, dr\, d\theta\, dz) \\ &= \rho \int_0^t \int_0^{2\pi} \frac{r^4}{4} d\theta\, dz \\ &= \rho \int_0^t \frac{r^4}{2} \pi\, dz = \rho \frac{r^4}{2} \pi\, t = \frac{r^2}{2} \rho\, \pi\, r^2\, t = \frac{r^2}{2} \rho V = \frac{r^2}{2} M \end{aligned}$$



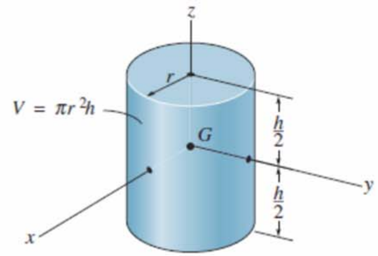
From inside back cover of Hibbler textbook

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



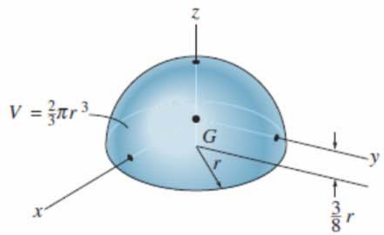
Sphere

$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m r^2$



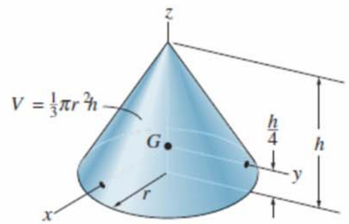
Cylinder

$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2) \quad I_{zz} = \frac{1}{2} mr^2$



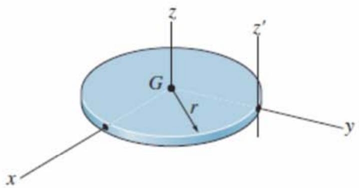
Hemisphere

$I_{xx} = I_{yy} = 0.259 mr^2 \quad I_{zz} = \frac{2}{5} mr^2$



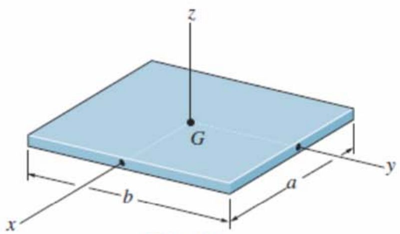
Cone

$I_{xx} = I_{yy} = \frac{3}{80} m(4r^2 + h^2) \quad I_{zz} = \frac{3}{10} mr^2$



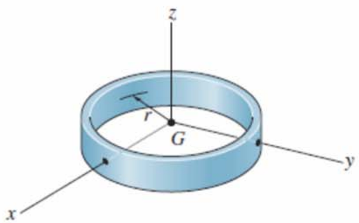
Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \quad I_{zz} = \frac{1}{2} mr^2 \quad I_{z'z'} = \frac{3}{2} mr^2$



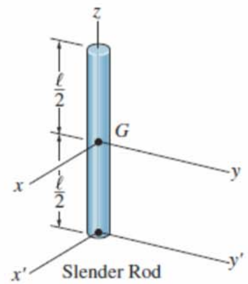
Thin plate

$I_{xx} = \frac{1}{12} mb^2 \quad I_{yy} = \frac{1}{12} ma^2 \quad I_{zz} = \frac{1}{12} m(a^2 + b^2)$



Thin ring

$I_{xx} = I_{yy} = \frac{1}{2} mr^2 \quad I_{zz} = mr^2$



Slender Rod

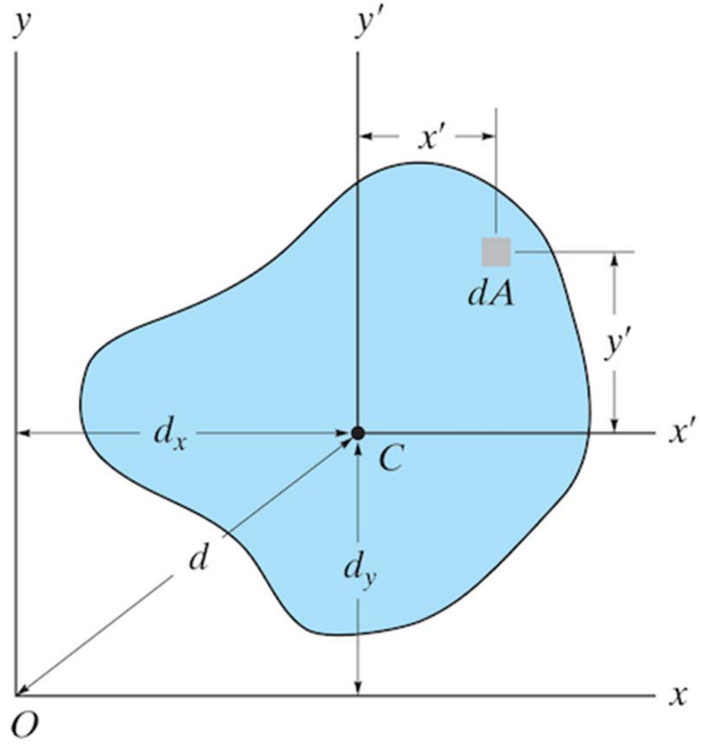
$I_{xx} = I_{yy} = \frac{1}{12} m\ell^2 \quad I_{x'x'} = I_{y'y'} = \frac{1}{3} m\ell^2 \quad I_{z'z'} = 0$

Recap: Parallel axis theorem

- Often, the **moment of inertia** of an area is known for an axis passing through the **centroid**; e.g., x' and y':
- The moments around other axes can be computed from the known $I_{x'}$ and $I_{y'}$:

$$I_x = I_{x'} + Ad_y^2$$
$$I_y = I_{y'} + Ad_x^2$$
$$J_O = J_C + A(d_x^2 + d_y^2) = J_C + Ad^2$$

Moments of inertia relative to axis through CENTROID ← Key Point!



Note: the integral over y' gives zero *when done through the centroid axis*.

Chapter 11: Virtual Work

Goals and Objectives

- Introduce the principle of virtual work
- Show how it applies to determining the equilibrium configuration of a series of pin-connected members

Methods to derive Equations of Equilibrium

- Force-Balance Method
  - $\sum \vec{\boldsymbol{F}} = 0, \sum \vec{\boldsymbol{M}} = 0$

- Work-Energy Method (or Virtual Work Method)
  - $\delta U = \sum \left(\vec{\boldsymbol{F}} \cdot \boldsymbol{\delta} \vec{\boldsymbol{r}} \right) + \sum \left(\vec{\boldsymbol{M}} \cdot \delta \vec{\boldsymbol{\theta}} \right) = 0$

- Virtual Work Method is particularly useful for structures with many members, whereas Force-Balance Method needs multiple equations per member

Aside: Recall from Physics: Energy, work and power

- Mechanical energy [joule (J)]:
  - Capacity of a body to do work
- Work [joule (J)]:
  - Energy change over a period of time
- Power [watt (W)]:
  - Rate at which work is done or energy is expended
- Joule = Watt * second

Aside: Mechanical energy [joule (J)]:

- Capacity of a body to do work
- Measure of the state of a body as to its ability to do work at an instant in time
- Kinetic energy:
  - Translational:
  - Rotational:
- Potential energy:
  - Gravitational:
  - Elastic:

$$KE_{trans} = \frac{1}{2} m v^2$$

$$KE_{rot} = \frac{1}{2} I_O \omega^2$$

$$PE_{grav} = mgh$$

$$PE_{elas} = \frac{1}{2} k x^2$$

Aside: Work [joule (J)]:

- Energy change over a period of time as a result of a force (or moment) acting through a translational (or rotational) displacement

$$U_{trans} = \int_{r1}^{r2} F\, dr \qquad U_{rot} = \int_{\theta 1}^{\theta 2} M\, d\theta$$

- Measure of energy flow from one body to another
  - Requires time to elapse
  - e.g., Energy flows from A to B ⟶ A does work on B

- Work generated by a force (or moment) is the dot product of the force and translational (rotational - angular) displacement at the point of application of the force

$$U_{trans} = \boldsymbol{F} \bullet \boldsymbol{r} \qquad U_{rot} = \boldsymbol{M} \bullet \boldsymbol{\theta}$$

Aside: Power [watt (W)]:

- Rate at which work is done or energy is expended

$$P = \frac{dU}{dt}$$

- Alternatively, work is the integral of power (area under the power curve)

$$W = \int_{t1}^{t2} P dt$$

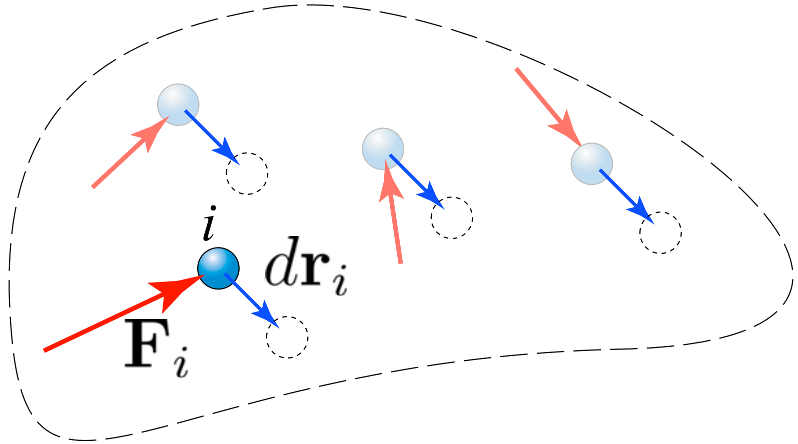
- Power generated by a force (or moment) is the dot product of the force and translational (rotational - angular) velocity at the point of application of the force

$$P_{trans} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$$

$$P_{rot} = \vec{\boldsymbol{M}} \cdot \vec{\boldsymbol{\omega}}$$

Definition of Work (U)

Work of a force



A force does work when it undergoes a displacement in the direction of the line of action.

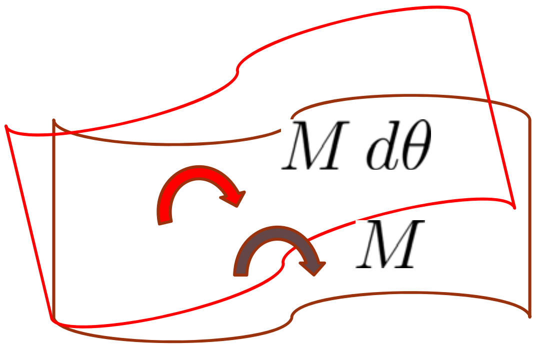
The work dU produced by the force $\boldsymbol{F}$ when it undergoes a differential displacement $d\boldsymbol{r}$ is given by

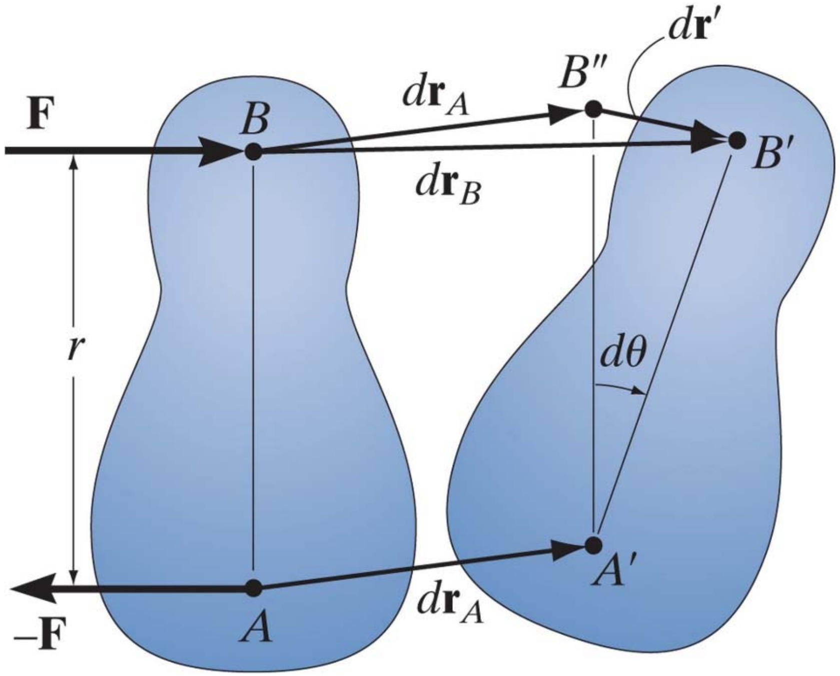
$$dU = \mathbf{F} \cdot d\mathbf{r}$$

Definition of Work (U)

Work of a couple moment

$$dU = M\mathbf{k} \cdot d\theta\,\mathbf{k} = M\,d\theta$$



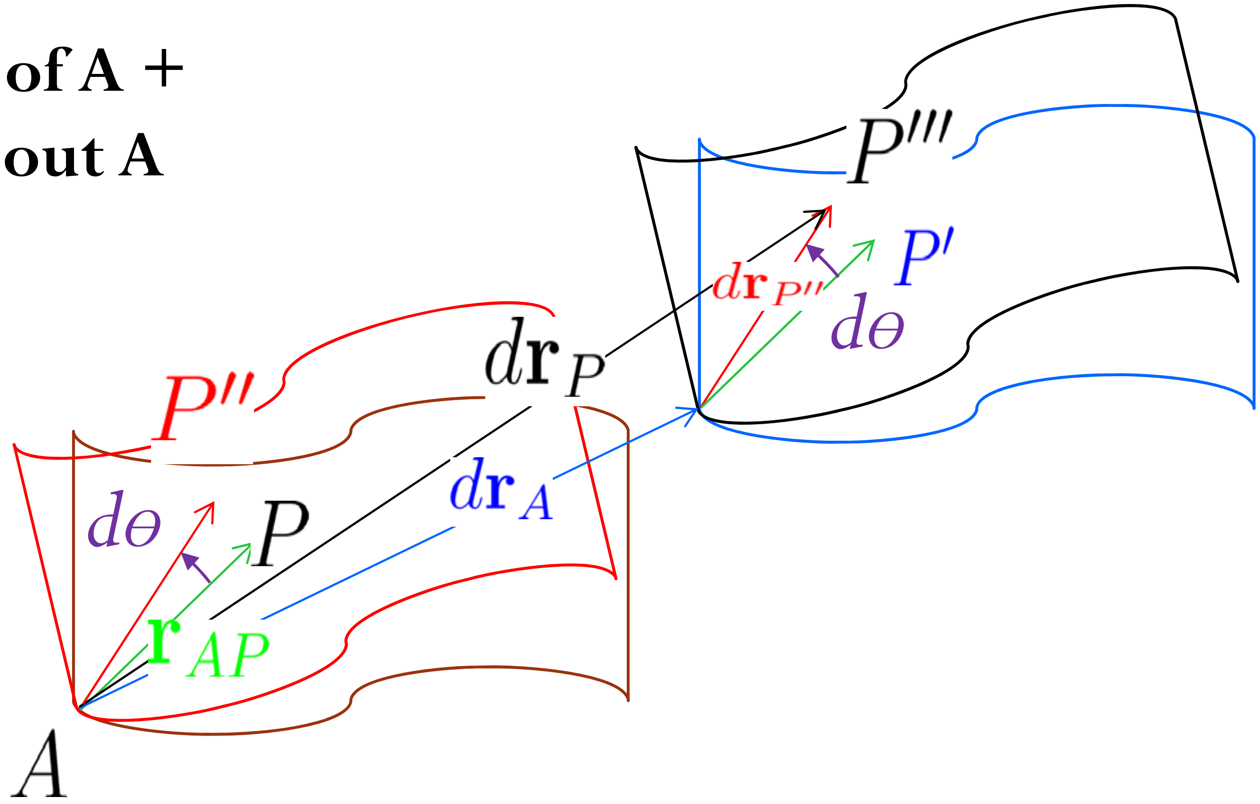


Incremental Displacement

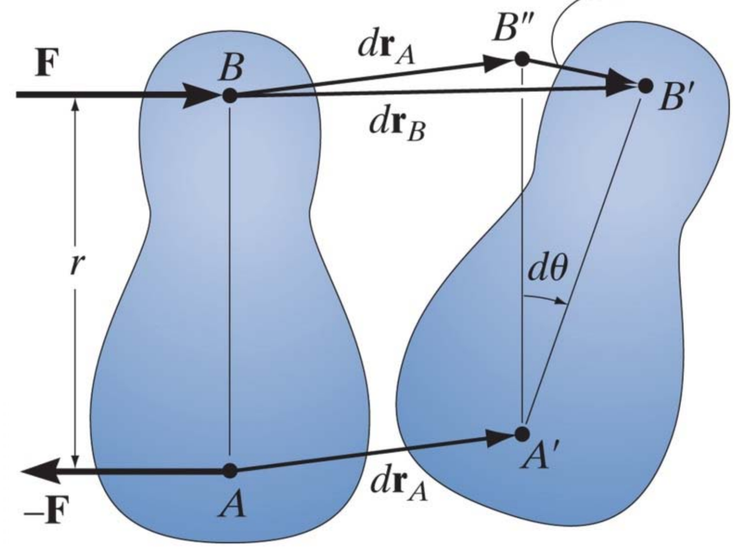
Rigid body displacement of P = translation of A + rotation about A

$$d\mathbf{r}_P = d\mathbf{r}_A + d\theta\,\mathbf{k} \times \mathbf{r}_{AP}$$

Translation of A +
Rotation about A



Definition of Work



Work of couple moment

$$d\mathbf{r}_P = d\mathbf{r}_A + d\theta\,\mathbf{k} \times \mathbf{r}_{AP}$$

$$\begin{aligned}
dU &= \sum_i \mathbf{F}_i \cdot d\mathbf{r}_i \\
&= \mathbf{F}_A \cdot d\mathbf{r}_A + \mathbf{F}_B \cdot d\mathbf{r}_B \\
&= -\mathbf{F} \cdot (d\mathbf{r}_A + d\theta\,\mathbf{k} \times \mathbf{r}_{AA}) + \mathbf{F} \cdot (d\mathbf{r}_A + d\theta\,\mathbf{k} \times \mathbf{r}_{AB}) \\
&= \mathbf{F} \cdot (d\theta\,\mathbf{k} \times \mathbf{r}_{AB}) \\
&= d\theta\,\mathbf{k} \cdot (\mathbf{r}_{AB} \times \mathbf{F}) \\
&= d\theta\,\mathbf{k} \cdot \mathbf{M}
\end{aligned}$$

$$\therefore\ dU = M\mathbf{k} \cdot d\theta\,\mathbf{k} = M\,d\theta$$

The couple forces do no work during the translation $d\boldsymbol{r}_A$

Work due to rotation

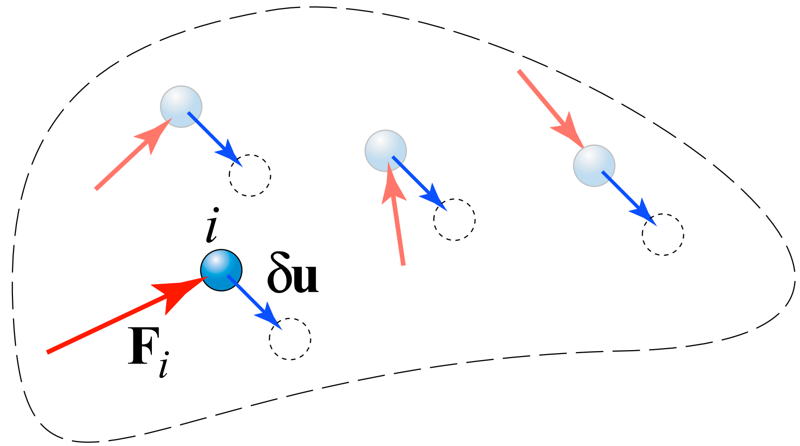
Virtual Displacements

A *virtual displacement* is a conceptually possible displacement *or* rotation of all *or* part of a system of particles. The movement is assumed to be possible, but actually does not exist. These "movements" are first-order differential quantity denoted by the symbol δ (for example, δr and $\delta\theta$.

Principle of Virtual Work

The principle of virtual work states that if a body is in equilibrium, then the algebraic sum of the virtual work done by all the forces and couple moments acting on the body is zero for any virtual displacement of the body. Thus,

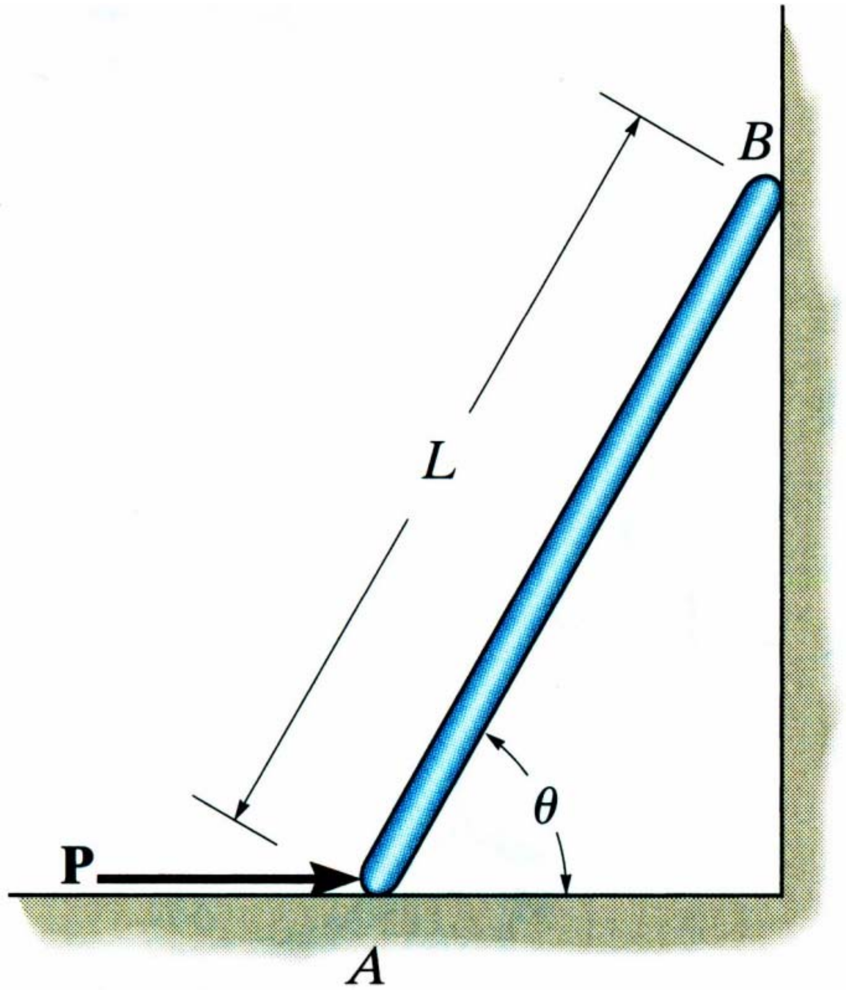
$$\delta U = 0$$



$$\delta U = \Sigma(\boldsymbol{F} \cdot \delta \boldsymbol{u}) + \Sigma(\boldsymbol{M} \cdot \delta \boldsymbol{\theta}) = 0$$

For 2D:

$$\delta U = \Sigma(\boldsymbol{F} \cdot \delta \boldsymbol{u}) + \Sigma(M\, \delta\theta) = 0$$

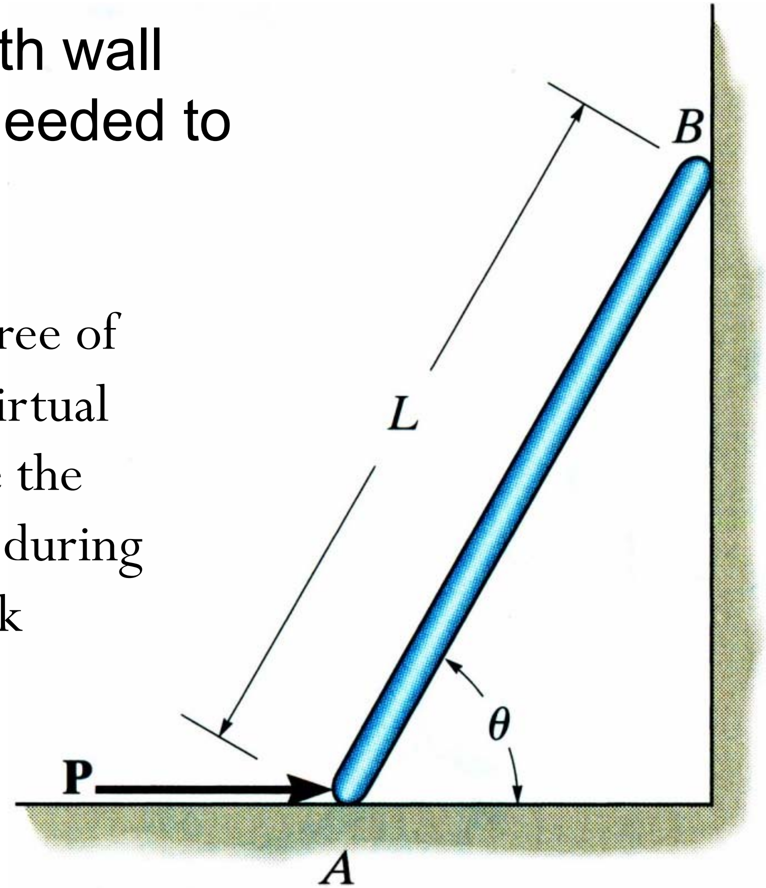


The thin rod of weight W rests against the smooth wall and floor. Determine the magnitude of force P needed to hold it in equilibrium.

Procedure for Analysis

1. Draw FBD of the entire system and provide coordinate system
2. Sketch the "deflected position" of the system
3. Define position coordinates measured from a fixed point and select the parallel line of action component and remove forces that do no work
4. Differentiate position coordinates to obtain virtual displacement
5. Write the virtual work equation and express the virtual work of each force/ couple moment
6. Factor out the comment virtual displacement term and solve

The thin rod of weight W rests against the smooth wall and floor. Determine the magnitude of force P needed to hold it in equilibrium.



Use the principle of virtual work. This problem has one degree of freedom, which we can take as the angle θ. Let $\delta\theta$ be the virtual rotation of the rod, such that the rod slides at A and B. Since the contact at A and B are smooth, the only forces that do work during the virtual displacements are P and W. Then the virtual work becomes:

Disk of 10 lb is subjected to a vertical force P = 8 lb and a couple moment M = 8 lb ft. Determine disk's rotation θ if the end of the spring wraps around the periphery of the disk as the disk turns. The spring is originally unstretched.