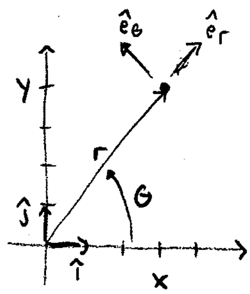
Lecture 2 — Announcements

Office Hours: Ray: M 3-4pm (after lecture) location TBA

Xian: T 10-12 MEB244

Bring a spreadsheet-capable device on Friday
HW1 available on PL

Brief Recap Cartesian and Polar coordinates and position vectors



P has position (x, y) or (r, θ)

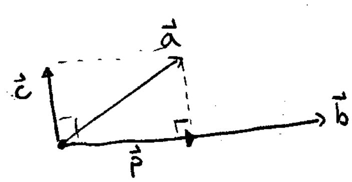
position vector $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$

$= r\,\hat{e}_r$

$\hat{e}_r = \cos\Theta\,\hat{\imath} + \sin\Theta\,\hat{\jmath}$ $\qquad$ $\hat{\imath} = \cos\Theta\,\hat{e}_r - \sin\Theta\,\hat{e}_\theta$

$\hat{e}_\theta = -\sin\Theta\,\hat{\imath} + \cos\Theta\,\hat{\jmath}$ $\qquad$ $\hat{\jmath} = \sin\Theta\,\hat{e}_r + \cos\Theta\,\hat{e}_\theta$

Decomposing vectors with projection



Break down $\vec{a}$ into two pieces

$\vec{a} = \vec{p} + \vec{c}$

- $\vec{p}$ is in the direction of $\vec{b}$
- $\vec{c}$ is perpendicular (orthogonal) to $\vec{b}$

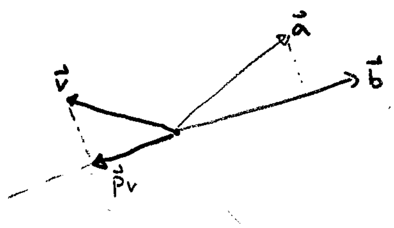
Orthogonal projection $\vec{p}$

- Direction: Same as $\vec{p}$ $\Rightarrow$ unit vector $\hat{p} = \dfrac{\vec{p}}{\|\vec{p}\|}$
- Magnitude: Geometry $\Rightarrow$ $\|\vec{a}\|\cos\Theta = \dfrac{\|\vec{a}\|\cdot\|\vec{b}\|}{\|\vec{b}\|}\cos\Theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} = \vec{a}\cdot\hat{b}$

So $\vec{p} = \underbrace{(\vec{a}\cdot\hat{b})}_{\text{Scalar projection}}\,\hat{b} = \dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}}{\|\vec{b}\|^2}$

$\vec{p} = \mathrm{Proj}(\vec{a}, \vec{b})$

"projection of $\vec{a}$ in the direction of $\vec{b}$"



Is the projection always the "same way" as $\vec{b}$? NO

If $\vec{v} \cdot \vec{b} < 0$, $\vec{p}_v$ points in the opposite direction

What if $\vec{v}$ and $\vec{b}$ are perpendicular? $\text{Proj}(\vec{a}, \vec{b}) = 0$

What is left over? Complementary projection

$$\vec{a} = \vec{p} + \vec{c} = \text{Proj}(\vec{a}, \vec{b}) + \text{Comp}(\vec{a}, \vec{b}) \Rightarrow \text{Comp}(\vec{a}, \vec{b}) := \vec{a} - \text{Proj}(\vec{a}, \vec{b})$$

If you know something about $\vec{a}, \vec{b}$ this can simplify

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Application: Change of basis

Suppose $\vec{r} = x\hat{\imath} + y\hat{\jmath} = r\hat{e}_r$ and $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}$

How to find $\vec{v}$ in the polar basis? $\vec{v} = \underbrace{(\vec{v} \cdot \hat{e}_r)\hat{e}_r}_{\downarrow} + (\vec{v} \cdot \hat{e}_\theta)\hat{e}_\theta$

$\text{Proj}(\vec{v}, \hat{e}_r)$ is the whole vector; $(\vec{v} \cdot \hat{e}_r)$ is a component

Can do dot products in any basis, as long as its the <u>same basis representation</u> for both

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What if I know $\vec{r}$ in two bases and not the relationship?

$$\vec{v} = x\hat{\imath} + y\hat{\jmath} = v_r\hat{e}_r + v_\theta\hat{e}_\theta$$ What is θ?

$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

Cartesian   Polar

$\boxed{Q}$ What is $\vec{v} \cdot \hat{e}_r$? $\boxed{x\cos\theta + y\sin\theta = v_r}$

What is $\vec{v} \cdot \hat{e}_\theta$?

Calculus on coordinates and vectors

Coordinates are scalars $\Rightarrow$ calculus works!

Position of P is $(x(t), y(t))$ Parametrized curve

Change in position is $\left(\frac{\partial x}{\partial t}(t), \frac{\partial y}{\partial t}(t)\right) = (\dot{x}(t), \dot{y}(t))$ Dots = derivative w.r.t. time

Q is on a curve at position $(x(t), y(x(t)))$

Change in position is $(\dot{x}(t), \dot{y}(t)) = (\dot{x}(t), y'(x(t))\dot{x}(t))$ Prime = derivative w.r.t. argument

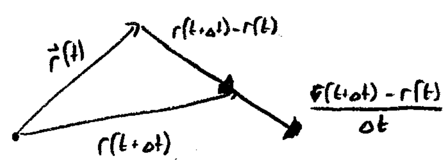
$\boxed{\text{CHAIN RULE}}$ Consider things time-varying unless you <u>know</u> they are constant

S is at $(r(\theta), \theta)$ change in position is $(r'(\theta)\dot{\theta}, \dot{\theta})$

<u>Vector derivatives</u>

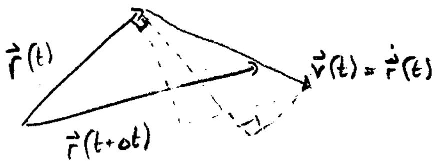
$$\dot{\vec{v}} = \frac{\partial}{\partial t}\vec{v} = \lim_{t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$

Graphical view



We could "approximate" $\dot{\vec{r}}$ by fixing Δt

How does $\vec{r}$ change in time?



Is $\vec{r}$ getting longer shorter Magnitude

turning clockwise or ccw Direction

$$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{v}(t)\Delta t$$

What part of $\vec{v}$ causes $\vec{r}$ to change length? $\vec{v}\cdot\hat{r}$ (This gives a rate of change)

Interpret $\vec{v}\cdot\hat{r}$ > 0 longer

$= 0$

< 0 shorter

What part of $\vec{v}$ causes $\vec{r}$ to change direction?

Derivatives of unit vectors

$\hat{a}$ is a unit vector $a = \|\hat{a}\| = 1$

$a^2 = \hat{a}\cdot\hat{a} = 1$

What is $\dot{a}$? 0

What is $\dot{\hat{a}}$? (How does it change direction?)

$$\frac{\partial}{\partial t}(\hat{a}\cdot\hat{a}) = \dot{\hat{a}}\cdot\hat{a} + \hat{a}\cdot\dot{\hat{a}} = 2\dot{\hat{a}}\cdot\hat{a} = \frac{\partial}{\partial t}(1) = 0$$

So $\dot{\hat{a}}$ is orthogonal to $\hat{a}$ Pure rotation

If $\vec{r} = r\hat{r}$ Then $\dot{\vec{r}} = \underbrace{\overbrace{\boxed{\dot{r}\hat{r}}}^{\text{Length changing}} + \boxed{r\dot{\hat{r}}}}_{}$ Direction changing

Q? Is $\dot{\hat{r}}$ necessarily a unit vector?

Q? Is $\|\dot{\vec{a}}\| = \dot{a}$