Lecture 4: Announcements

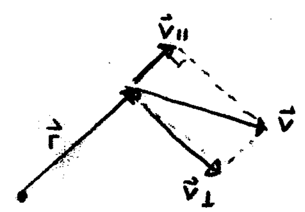
HW 1 Due Wednesday

Office Hours: After class today in 252 MEB

Project Checkpoint 2(a), (b) due week from Wednesday

Recap:

The derivative of a vector has a length-changing part and a direction-changing part



$$\dot{\vec{r}} = \underbrace{\dot{r}\,\hat{r}}_{\substack{\vec{v}_{\parallel} \\ \text{Length-change}}} + \underbrace{r\,\dot{\hat{r}}}_{\substack{\vec{v}_{\perp} \\ \text{Direction-change}}} \qquad \hat{r} \perp \dot{\hat{r}}$$

Derivatives in polar basis

If $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ then $\vec{v} = \dot{\vec{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}\ \left(+\cancel{x\dot{\hat{\imath}} + y\dot{\hat{\jmath}}}\right)^{0}$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath}$$

In polar $\vec{r} = r\hat{e}_r$ so $\vec{v} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$ $\qquad \hat{e}_r$ changes in time if P moves

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\dot{\hat{e}}_r = -\dot{\theta}\sin\theta\,\hat{\imath} + \dot{\theta}\cos\theta\,\hat{\jmath}$$

$$= \dot{\theta}\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)$$

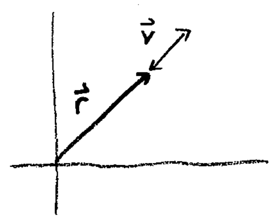
$$= \dot{\theta}\,\hat{e}_\theta$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

Some sample motions: $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$

$\dot{\theta} = \omega$ angular velocity

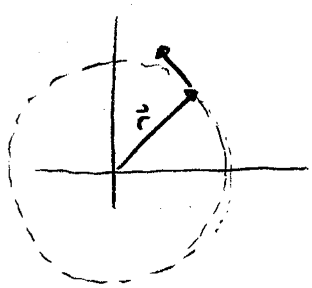
$\dot{\theta} = 0\,,\ \dot{r} \neq 0$



$\vec{v} = \dot{r}\,\hat{e}_r$

Only length changing - linear motion

$\dot{\theta} \neq 0\,,\ \dot{r} = 0$

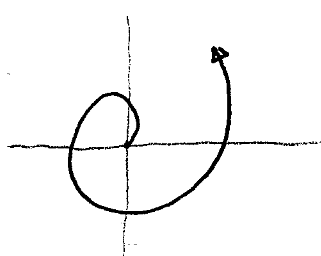


$\vec{v} = r\dot{\theta}\,\hat{e}_\theta$

Only direction-changing

Uniform circular motion

$\dot{\theta} \neq 0\,,\ \dot{r} \neq 0$



$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$

Spiral motion

Accelerations

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a} = \dot{\vec{v}} = \left[\ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r\right] + \left[\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta\right]$$

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\cos\theta\,\hat{\imath} - \dot{\theta}\sin\theta\,\hat{\jmath}$$

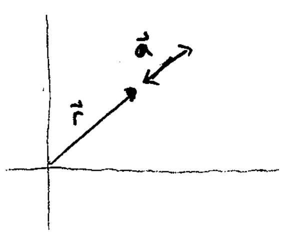
$$= -\dot{\theta}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right)$$

$$= -\dot{\theta}\,\hat{e}_r$$

$$\vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r$$

$$= \left[\ddot{r} - r\dot{\theta}^2\right]\hat{e}_r + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{e}_\theta$$

Simpler Motions

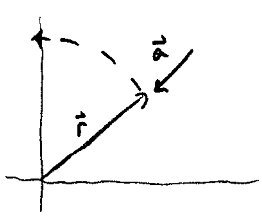


$\vec{a} = \ddot{r}\,\hat{e}_r$

What is constant?

θ $\dot{\theta}$ r $\dot{r}$

Linear motion - radial acceleration

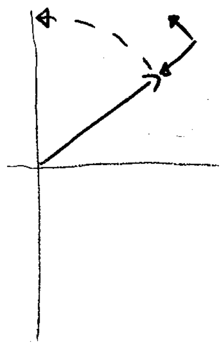


$\vec{a} = -r\dot{\theta}^2\,\hat{e}_r$

What is constant?

θ $\dot{\theta}$ r $\dot{r}$

Uniform circular motion - centripedal acceleration



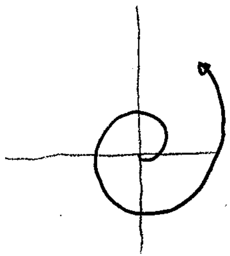
$\vec{a} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$

What is constant?

$\theta \quad \dot{\theta} \quad r \quad \dot{r}$

~~Uniform~~ circular motion with changing speed

Angular acceleration



$\vec{a} = -r\dot{\theta}^2 + 2\dot{r}\dot{\theta}\hat{e}_\theta$

What is constant?

$\theta \quad \dot{\theta} \quad r \quad \dot{r}$

Spiraling

Names of acceleration terms

$$\vec{a} = [\underbrace{\ddot{r}}_{\text{radial}} - \underbrace{r\dot{\theta}^2}_{\text{centripetal}}]\hat{e}_r + [\underbrace{2\dot{r}\dot{\theta}}_{\text{Coriolis}} + \overbrace{r\ddot{\theta}}^{\text{angular}}]\hat{e}_\theta$$

centripetal "center-seeking"

	Cartesian	Polar
Position	$x\hat{\imath} + y\hat{\jmath}$	$r\hat{e}_r$
Velocity	$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$	$\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
Acceleration	$\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$	$(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$

Integrating in Polar (coordinates)

Cartesian integration is not so bad

$$\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$$

$$\vec{r} = \int \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}\ \partial t = \hat{\imath}\int \dot{x}\,\partial t + \hat{\jmath}\int \dot{y}\,\partial t$$ $\hat{\imath}$ and $\hat{\jmath}$ are constant

What about in polar?

- Integrating polar vectors is very hard (even if you substitute)

Soln: Integrate components as coordinates

$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

Radial coordinate $\quad r(t) = \int_0^t \dot{r}(\tau)\,\partial\tau + r(0) = \int_0^t v_r\,\partial\tau + r(0)$

Angular coordinate $\quad \theta(t) = \int_0^t \dot{\theta}(\tau)\,\partial\tau + \theta(0) = \int_0^t \frac{v_\theta(\tau)}{r(\tau)}\,\partial\tau + \theta(0)$

(Second one is harder)

Example $\quad P(0) = \left(2, \frac{\pi}{4}\right) \qquad \vec{v} = (1)\,\hat{e}_r + (2)\,\hat{e}_\theta \qquad P(3) = ?$

$$r(t) = \int_0^t 1\,\partial\tau + 2 = t + 2 \Rightarrow r(3) = 5$$

$$\theta(t) = \int_0^t \frac{2}{r(\tau)}\,\partial\tau + \theta(0) = \int_0^t \frac{2}{t+2}\,\partial\tau + \theta(0)$$

$$= 2\ln(t+2) + \frac{\pi}{4}$$

$$\theta(3) \approx 4.004$$