$\boxed{\text{Lecture 5}}$ Announcements

HW1 Due Today

Quiz 1 Today – Friday

HW2 Available Today

Recap

	Cartesian	Polar
Position	$x\hat{\imath} + y\hat{\jmath}$	$r\hat{e}_r$
Velocity	$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$	$\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
Acceleration	$\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$	$(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$

Derivatives of polar basis vectors

$$\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$$

Intuition of angular velocity

The above show direction-change for unit vectors

The rate of change is $\dot{\theta}$

The direction is perpendicular to

- The original unit vector
- The <u>axis</u> of rotation

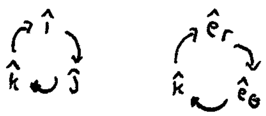
Angular velocity definition

We will define $\vec{\omega}$ as the angular velocity vector for the unit vector $\hat{a}$

$\vec{\omega}$ such that $\quad \dot{\hat{a}} = \vec{\omega} \times \hat{a}$

Example: For $\hat{e}_r$, $\vec{\omega} = \dot{\theta}\hat{k}$ s.t. $\vec{\omega} \times \hat{e}_r = (\dot{\theta}\hat{k}) \times \hat{e}_r = \dot{\theta}\,\hat{e}_\theta$

Side note:



Right-handed basis systems

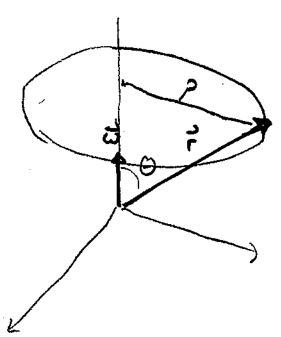
Then more generally

$$
\begin{aligned}
\dot{\vec{a}} &= \dot{a}\hat{a} + a\dot{\hat{a}} \\
&= \dot{a}\hat{a} + a(\vec{\omega} \times \hat{a}) \\
&= \dot{a}\hat{a} + \vec{\omega} \times (a\hat{a}) \\
&= \underbrace{\dot{a}\hat{a}}_{\text{Length-change}} + \underbrace{\vec{\omega} \times \vec{a}}_{\text{Direction-change}}
\end{aligned}
$$

* In 2D, $\omega = \dot{\theta}$

This isn't true in more arbitrary set-ups

A 3-D example



If the plane containing the path contained the origin

$v = \rho\omega = (r\sin\theta)\,\omega$

What is $\vec{v}$? $\quad \vec{v} = \vec{\omega} \times \vec{r}$

$|\vec{v}| = \omega r \sin\theta$ ✓

Example $\quad \vec{r} = 4\hat{\imath} + 3\hat{k}$ m $\quad$ What is $\vec{v}$?

$\vec{\omega} = 3\hat{k}$ rad/s

$$\vec{v} = \vec{\omega} \times \vec{r} = (3\hat{k}) \times (4\hat{\jmath} + 3\hat{k})$$

$$= (3\hat{k} \times 4\hat{\jmath}) + (3\hat{k} \times 3\hat{k})$$

$$= -12\hat{\imath}$$

Example $\quad$ A particle at $\vec{r} = 4\hat{\imath} + 2\hat{\jmath}$ m has speed $v = 8$ m/s

Which angular velocities are consistent with this?

$\vec{\omega} = 2\hat{\jmath}$ rad/s $\quad \vec{\omega} = 2\hat{\imath}$ rad/s $\quad \vec{\omega}_s = -4\hat{\imath}$ rad/s

$\vec{\omega} = 4\hat{\jmath}$ rad/s $\quad \vec{\omega} = 4\hat{\imath}$ rad/s

$\vec{v} =$

Derivatives and angular accelerations

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{a} = \dot{\vec{v}} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}$$

$$= \boxed{\vec{\alpha}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times r)$$ These parenthesis matter!

A note on planar cross products

Suppose $\vec{r} = r_x \hat{\imath} + r_y \hat{\jmath}$ $\quad \vec{\omega} = \omega \hat{k}$

Then $\vec{v} = (\omega \hat{k}) \times (r_x \hat{\imath} + r_y \hat{\jmath})$

$$= (\omega \hat{k} \times r_y \hat{\jmath}) + (\omega \hat{k} \times r_x \hat{\imath})$$

$$= -\omega r_y \hat{\imath} + \omega r_x \hat{\jmath}$$

$$= \omega(-r_y \hat{\imath} + r_x \hat{\jmath})$$

$$= \omega \vec{r}^{\perp}$$

* If $\vec{r} = [r_x, r_y, 0]$ $\quad \vec{r}^{\perp} = [-r_y, r_x, 0]$

Note $\vec{r} \cdot \vec{r}^{\perp} = -r_x r_y + r_y r_x = 0$ ✓

Notice $\|\vec{r}^{\perp}\| = \|\vec{r}\|$

There's one other vector with these properties $[r_y, -r_x, 0]$

$\vec{r}^{\perp}$ is the counter-clockwise rotation ; $-\vec{r}^{\perp}$ is clockwise

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Summary Rotating about axis with fixed-length $\vec{r}$

Position $\vec{r}$

Velocity $\vec{v} = \vec{\omega} \times \vec{r}$

Acceleration $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$