$\boxed{\text{Lecture 6}}$ Announcements

Project Checkpoint 2(a), (b) due next Wed

No class Fri, 6/29, or Wed 7/4

Recap Describing motions with vectors

	Cartesian	Polar
Position	$x\hat{\imath} + y\hat{\jmath}$	$r\hat{e}_r$
Velocity	$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$	$\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$
Acceleration	$\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$	$(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$

* Both of these bases require the existence of some origin and axis

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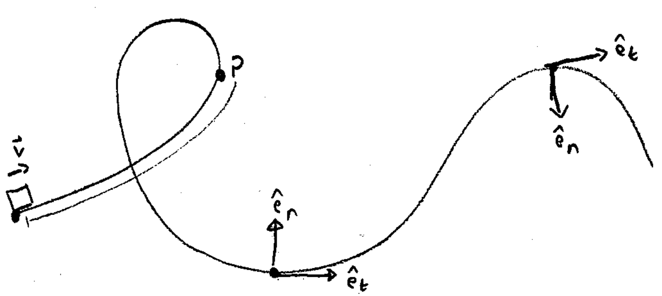
A complicated trajectory



There's no obviously good place to put axis, origin

Distance traveled along the path = arclength = s

Speed = $v = \dot{s}$

Let $\boxed{\hat{e}_t := \frac{\vec{v}}{v}}$ Direction of travel

Then $\vec{v} = \dot{s}\,\hat{e}_t$

Build a normal vector $\hat{e}_n = \dfrac{\dot{\hat{e}}_t}{|\dot{\hat{e}}_t|}$:= (so $\hat{e}_n \perp \hat{e}_t$)

The tangential-normal basis: $\hat{e}_t$ Direction of motion

$\hat{e}_n$ Direction of turning

Another way of expressing $\hat{e}_n$

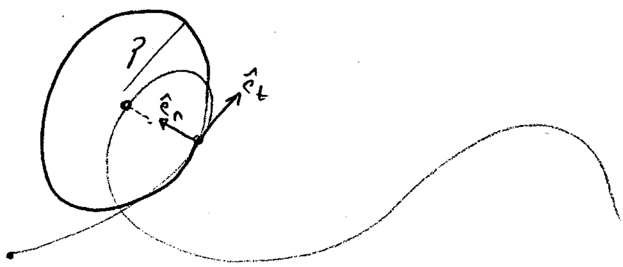
$$\vec{v} = \dot{s}\,\hat{e}_t$$

$$\vec{a} = \ddot{s}\,\hat{e}_t + \dot{s}\,\dot{\hat{e}}_t$$ Direction of turning

Then $Proj(\vec{a}, \hat{e}_t) = \ddot{s}\,\hat{e}_t$ and $Comp(\vec{a}, \hat{e}_t) = \vec{a} - \ddot{s}\,\hat{e}_t = \dot{s}\,\dot{\hat{e}}_t$

So $\hat{e}_n = \dfrac{\dot{\hat{e}}_t}{|\dot{\hat{e}}_t|} = \dfrac{Comp(\vec{a}, \hat{e}_t)}{|Comp(\vec{a}, \hat{e}_t)|}$

Another physical intuition for this trajectory



The osculating circle is the circle tangent to the trajectory "on the inside" which matches the 1st and 2nd derivative

ρ = radius of curvature = radius of osculating circle

κ = kappa = curvature = $\dfrac{1}{\rho}$

Small $\rho \Rightarrow$ big κ "curvy" Large $\rho \Rightarrow$ small κ "not curvy" (almost straight)

How does this relate to $\dot{\hat{e}}_t$?

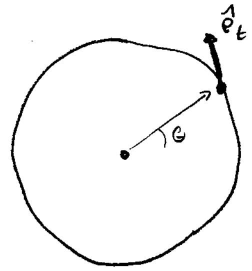
$$\dot{\hat{e}}_t = \frac{\partial}{\partial t}\hat{e}_t = \frac{\partial}{\partial s}\hat{e}_t \frac{\partial s}{\partial t} = \dot{s}\boxed{\frac{\partial}{\partial s}\hat{e}_t}$$ How does direction of $\hat{e}_t$ change as you move along the curve?

$$= \dot{s}\,\kappa\,\hat{e}_n$$

Then acceleration is $\vec{a} = \ddot{s}\,\hat{e}_t + \dot{s}\,\dot{\hat{e}}_t = \ddot{s}\,\hat{e}_t + \dot{s}\,(\dot{s}\,\kappa\,\hat{e}_n) =$

$$\vec{a} = \ddot{s}\,\hat{e}_t + \dot{s}^2\,\kappa\,\hat{e}_n$$

Example Uniform circular motion



Relate TN and polar for this example

$$\hat{e}_t = + \quad - \qquad \hat{e}_r \quad \hat{e}_\theta$$

$$\hat{e}_n = + \quad - \qquad \hat{e}_r \quad \hat{e}_\theta$$

$$\vec{a} = (\cancelto{0}{\ddot{r}} - r\dot{\theta}^2)\,\hat{e}_r + (\cancelto{0}{2\dot{r}\dot{\theta}} + \cancelto{0}{r\ddot{\theta}})\,\hat{e}_\theta$$

$$= -r\dot{\theta}^2\,\hat{e}_r$$

$$= \frac{-(r\dot{\theta})^2\,\hat{e}_r}{r}$$

$$= \frac{-v^2}{r}\,\hat{e}_r = \boxed{\frac{\dot{s}^2}{r}\,\hat{e}_n} \Rightarrow \boxed{\kappa = \frac{1}{r}}$$

What is $\ddot{s}$?

Acceleration — Rate of change of velocity

Magnitude of acceleration — Rate of change of speed