Lecture 9 | Announcements

No lecture or OH on Wednesday (7/4) - Piazza open

HW3 Due Wed - Q3 Thurs - Fri

Proj checkpoint 2(c) on Friday

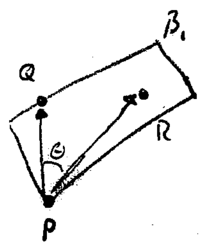
From particles to rigid bodies

* Rigid body assumption: Distances between points on the body are constant

r_{PQ} = constant $\qquad \vec{r}_{PQ}$ constant? NO

Can rotate!

How does the direction change?



$r_{PQ} = \text{const} \qquad \vec{r}_{PQ}$ can rotate

$r_{PR} = \text{const} \qquad \vec{r}_{PR}$ can rotate

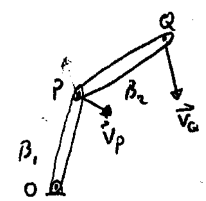
How do we handle rotating vectors of constant length?

$$\dot{\vec{r}}_{PQ} = \cancel{\dot{r}_{PQ}\hat{r}_{PQ}} + \vec{\omega}_1 \times \vec{r}_{PQ} \qquad \dot{\vec{r}}_{PR} = \vec{\omega}_2 \times \vec{r}_{PR}$$

How are $\vec{\omega}_1$ and $\vec{\omega}_2$ related? Do $\vec{r}_{PQ}$ and $\vec{r}_{PR}$ turn at the same speed?

Conclusion: $\vec{\omega}$ is a property of the <u>whole body</u>, not points

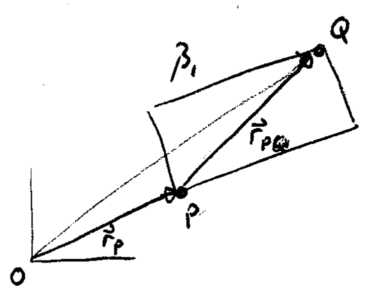
* Position, velocity are properties of individual points



$\vec{\omega}_1$ with β_1 $\qquad$ $\vec{v}_P$ with P

$\vec{\omega}_2$ with β_2 $\qquad$ $\vec{v}_Q$ with Q

Rigid Body velocity formula



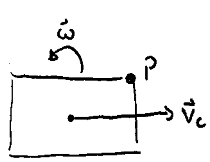
$$\vec{r}_Q = \vec{r}_{OQ} = \vec{r}_P + \vec{r}_{PQ}$$

$$\vec{v}_Q = \dot{\vec{r}}_Q = \dot{\vec{r}}_P + \dot{\vec{r}}_{PQ}$$

$$= \vec{v}_P + \vec{\omega}_1 \times \vec{r}_{PQ}$$

P and Q are <u>any</u> points on the body

<u>Example</u>

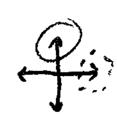


$\vec{v}_c = 3\hat{\imath}$ m/s

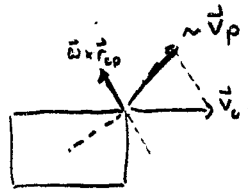
$\vec{\omega} = 2\hat{k}$ rad/s

$\vec{r}_{cP} = (2\hat{\imath} + \hat{\jmath})$ m

What is $\vec{v}_P$?

Predict: Closest cardinal direction to $\vec{v}_P$? 

Graphical



Compute $\vec v_p = \vec v_c + \vec\omega \times \vec r_{cp}$

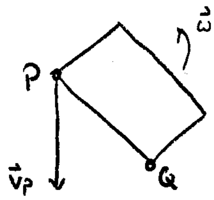
$$= (3\hat i) + (2\hat k) \times (2\hat i + \hat j)$$

$$= 3\hat i + 4\hat j - 2\hat i$$

$$= \hat i + 4\hat j$$

Check: Is this reasonable? Is this consistent?

<u>Example</u>



$\vec v_p = -3\hat j$

$\vec r_{PQ} = (2\hat i - 2\hat j)$

$w = \hat k$

Predict? $\longrightarrow$

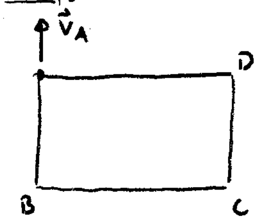
Compute $\vec v_Q = -3\hat j + \hat k \times (2\hat i - 2\hat j)$

$$= -3\hat j + 2\hat j + 2\hat i$$

$$= -\hat j + \hat i$$

Reflect

Example



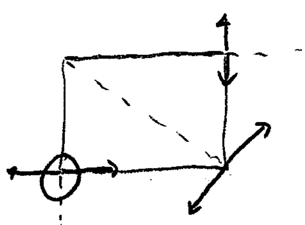
V_Q is 

Which point is Q?

$\vec{V}_A + \omega \times r_{AQ} = V_Q$

$\uparrow + ? = \nearrow$

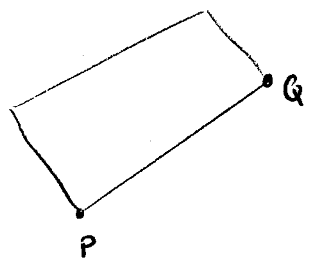
So $\vec{\omega} \times r_{AQ}$ is $\longrightarrow$ What is consistent with this?



Direction of $\vec{\omega}$ affects answer

Only B is consistent

Rigid Body Accelerations



$$\vec{v}_Q = \vec{v}_P + \vec{\omega}_1 \times \vec{r}_{PQ}$$

$$\dot{\vec{v}}_Q = \dot{\vec{v}}_P + \frac{\partial}{\partial t}\left(\vec{\omega}_1 \times \vec{r}_{PQ}\right)$$

$$\boxed{\vec{a}_Q = \vec{a}_P + \vec{\alpha}_1 \times \vec{r}_{PQ} + \vec{\omega}_1 \times \left(\vec{\omega}_1 \times \vec{r}_{PQ}\right)}$$

Example Pendulum

$$\vec{r}_{PQ} = 3\hat{\imath} - 4\hat{\jmath} \quad m$$

$$\vec{\omega} = 2\hat{k} \quad rad/s$$

$$\vec{\alpha} = -\hat{k} \quad rad/s^2$$

$$\vec{v}_Q = \vec{v}_P + \vec{\omega} \times \vec{r}_{PQ}$$

$$= 0 + (2\hat{k}) \times (3\hat{\imath} - 4\hat{\jmath}) = 8\hat{\imath} + 6\hat{\jmath} \quad m/s$$

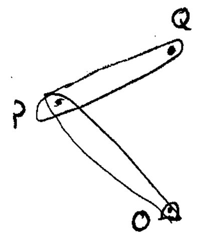
$$\vec{a}_Q = \cancelto{0}{\vec{a}_P} + (-\hat{k}) \times (3\hat{\imath} - 4\hat{\jmath}) + (2\hat{k}) \times \left[(2\hat{k}) \times (3\hat{\imath} - 4\hat{\jmath})\right]$$

$$-4\hat{\imath} - 3\hat{\jmath} + (2\hat{k}) \times (8\hat{\imath} + 6\hat{\jmath})$$

$$-4\hat{\imath} - 3\hat{\jmath} - 12\hat{\imath} + 16\hat{\jmath}$$

$$= -16\hat{\imath} + 13\hat{\jmath}$$

Example: Double Pendulum (follow on)



$\vec{r}_{OP} = -2\hat{i} + 2\hat{j}$

$\vec{r}_{PQ} = 2\hat{i} + \hat{j}$

$\vec{v}_Q = 7\hat{i} + 4\hat{j}$

$\vec{a}_Q = 19\hat{i} - 19\hat{j}$

What are the angular velocities $\omega_1, \omega_2 ; \alpha_1, \alpha_2$?

$$\begin{aligned}\vec{v}_P &= \omega_1 \hat{k} \times (-2\hat{i} + 2\hat{j}) = -2\omega_1 \hat{i} - 2\omega_1 \hat{j} \\ &= \vec{v}_Q + \omega_2 \hat{k} \times (-2\hat{i} - \hat{j}) \\ &= 7\hat{i} + 4\hat{j} - 2\omega_2 \hat{j} + \omega_2 \hat{i}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}-2\omega_1 &= 7 + \omega_2 \\ -2\omega_1 &= 4 - 2\omega_2 \\ 0 &= 3 + 3\omega_2 \qquad \omega_2 = -1\end{aligned}$$

$$\Rightarrow \boxed{\omega_1 = -3}$$

$$\begin{aligned}\vec{a}_P &= \alpha_1 \hat{k} \times (-2\hat{i} + 2\hat{j}) - (-3)^2(-2\hat{i} + 2\hat{j}) \\ &= -2\alpha\hat{i} - 2\alpha\hat{j} + 18\hat{i} - 18\hat{j} \\ &= (19\hat{i} - 19\hat{j}) + \alpha_2 \hat{k} \times (-2\hat{i} - \hat{j}) - (-1)^2(-2\hat{i} - \hat{j}) \\ &\quad 21\hat{i} - 18\hat{j} + \alpha_2 \hat{i} - 2\alpha_2 \hat{j}\end{aligned}$$

$$\begin{aligned}-2\alpha_1 + 18 &= 21 + \alpha_2 \\ -2\alpha_1 - 18 &= -18 - 2\alpha_2 \\ 36 &= 39 + 3\alpha_2 \Rightarrow \boxed{\alpha_2 = -1} \\ -2\alpha_1 + 18 &= 20 \Rightarrow \boxed{\alpha_1 = -1}\end{aligned}$$