Rigid Body Kinetics

For particles: $\sum \vec{F} = m\vec{a}$

For rigid bodies:

$$\boxed{\begin{aligned} \sum \vec{F} &= m\vec{a}_c \\ \sum \vec{M}_c &= I_c \vec{\alpha} \end{aligned}}$$

OR

$$\boxed{\sum \vec{M}_o = I_o \alpha}$$

$$\boxed{\begin{array}{c}\text{Forces} \\ \text{Moments}\end{array}} \xleftrightarrow[m,\ I]{\text{inertia}} \boxed{\begin{array}{c}\text{Motion} \\ \vec{a},\ \vec{\alpha}\end{array}}$$

Same two approaches

1. Assumed forces: $\vec{F}, \vec{M} \rightarrow \vec{a}, \vec{\alpha}$ (simulation)
2. Assumed motion: $\vec{a}, \vec{\alpha} \rightarrow \vec{F}, \vec{M}$ (measurement)

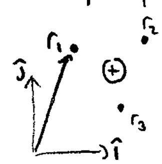
Same approach

1. Diagram and coordinates
2. Free body diagrams - $\vec{F}, \vec{M}$
3. Kinematics - $\vec{a}, \vec{\alpha}$
4. Kinetics - $\sum \vec{F} = m\vec{a}_c$ and $\sum \vec{M}_c = I_c \alpha$
5. Algebra

To handle this approach, we need to understand <u>mass</u> and <u>inertia</u>

$\boxed{\text{Mass}}$ One particle has mass m_i

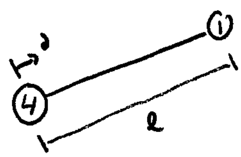
Multiple particles : Center of Mass (average location of system mass)



M = Total mass $= \sum m_i$

$$\vec{r}_{com} = \frac{1}{M} \sum m_i \vec{r}_i$$

Example (2 mass)



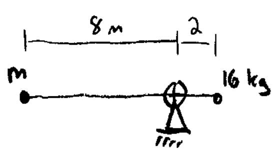
Position of com $\frac{?}{\ell}$

$$r_c = \frac{1}{M} \sum m_i r_i$$

$$= \frac{1}{(4+1)} (0 \cdot 4 + \ell \cdot 1)$$

$$= \boxed{\frac{\ell}{5}}$$

Example (2 masses)

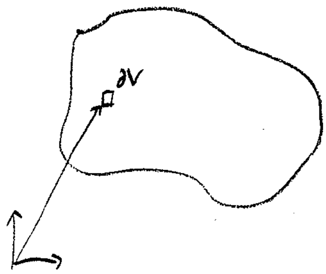


What is m?

$$0 = \frac{1}{m+16} (2 \cdot 16 - 8 \cdot m)$$

$\boxed{m = 4}$

What if we have a <u>lot</u> of little particles?



Tiny volume ∂V

Density ρ

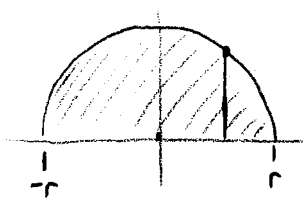
$\Big]$ Tiny mass $\partial m = \boxed{\rho \, \partial V}$

$\cancel{\Sigma}$

Total mass: $M = \int_V \partial m = \iiint_V \rho \, \partial V$

Center of mass: $\vec{r}_c = \frac{1}{M}\int_V \vec{r}_i \, \partial m = \frac{1}{M}\iiint_V \vec{r} \, \rho \, \partial V$

$\boxed{\text{Example}}$ Half disk Uniform area density



A point in the disk has coordinates (x, y)

$-1 \le x \le 1 \qquad 0 \le y \le \sqrt{r^2 - x^2}$

$\partial V = \partial y \, \partial x$

$= r \, \partial r \, \partial \theta$

$$M = \int_{-1}^{1} \int_0^{\sqrt{r^2-y^2}} \rho \, \partial y \, \partial x$$

$$= \int_{-1}^{1} \rho y \Big|_0^{\sqrt{r^2-x^2}} \partial x$$

$$= \int_{-1}^{1} \rho \sqrt{r^2 - x^2} \, \partial x$$

...

$$M = \int_0^{\pi} \int_0^{r} \rho R \, \partial R \, \partial \theta$$

$$= \int_0^{\pi} \frac{1}{2}\rho R^2 \Big|_0^{r} \partial \theta$$

$$= \int_0^{\pi} \frac{1}{2}\rho r^2 \, \partial \theta$$

$$= \boxed{\frac{1}{2}\rho \pi r^2}$$

Center of mass

$$\vec{r}_c = \frac{1}{M}\int_{-1}^{1}\int_{0}^{\sqrt{r^2-x^2}} \rho(x\hat{\imath} + y\hat{\jmath})\,\partial y\,\partial x$$

$$= \frac{1}{M}\int_{-1}^{1} \rho\left(x\,y\hat{\imath} + \tfrac{1}{2}y^2\hat{\jmath}\right)\Big|_0^{\sqrt{r^2-x^2}}\,\partial x$$

$$= \frac{1}{M}\int_{-1}^{1} \rho\left(x\sqrt{r^2-x^2}\,\hat{\imath} + \tfrac{1}{2}(r^2-x^2)\hat{\jmath}\right)\partial x$$

$$(r^2-x^2)^{\frac{1}{2}}x \;=\; -\tfrac{3}{4}(r^2-x^2)^{\frac{1}{2}}x\left(\tfrac{-2}{3}\right)$$

$$\frac{\partial}{\partial t}(r^2-x^2)^{\frac{3}{2}}\left(-\tfrac{1}{3}\right)$$

$$= \frac{1}{M}\left(\rho\left(-\tfrac{1}{3}(r^2-x^2)^{\frac{3}{2}}\hat{\imath} + \tfrac{1}{2}r^2x\hat{\jmath} - \tfrac{1}{6}x^3\hat{\jmath}\right)\Big|_{-r}^{r}\right)$$

$$\frac{1}{M}\left(\quad 0\,\hat{\imath} \quad + r^3\hat{\jmath} - \tfrac{1}{3}r^3\right.$$

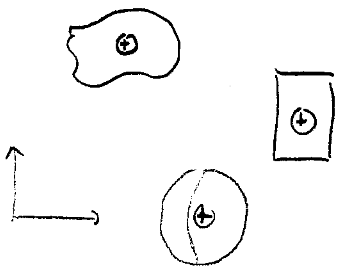
$$= \frac{1}{\left(\frac{1}{2}\pi r^2\right)}\left(\tfrac{2}{3}r^3\right)$$

$$= \boxed{\frac{4}{3}\frac{r}{\pi}}$$

For Common shapes: tables

For uncommon shapes: Suspension

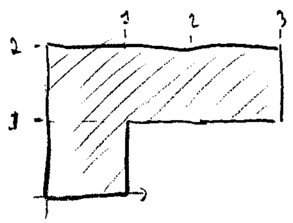
COM for composite bodies



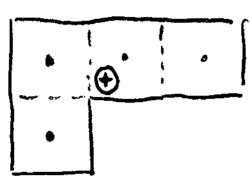
$$m = \sum m_i$$

$$\vec{r}_c = \frac{1}{m} \sum \vec{r}_c$$

Example



Constant com. Where is it?



$$\vec{r}_c = \frac{1}{4}\left[(.5\hat{\imath} + .5\hat{\jmath}) + (0.5\hat{\imath} + 1.5\hat{\jmath}) + (1.5\hat{\imath} + 1.5\hat{\jmath}) + (2.5\hat{\imath} + 1.5\hat{\jmath}) \right]$$

$$= \frac{1}{4}\left[5\hat{\imath} + 5\hat{\jmath} \right]$$

$$= 1.25\hat{\imath} + 1.25\hat{\jmath}$$

Q Is the COM always inside the body? In the convex hull of body? ✓