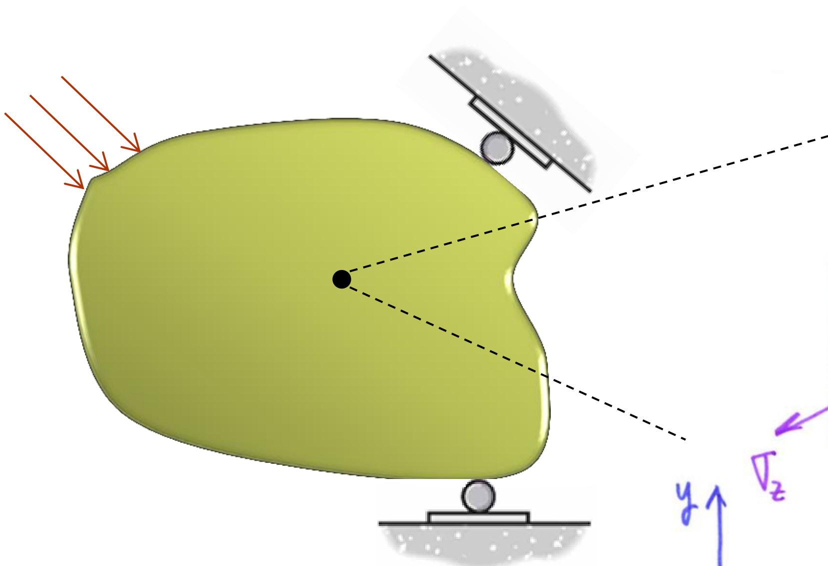
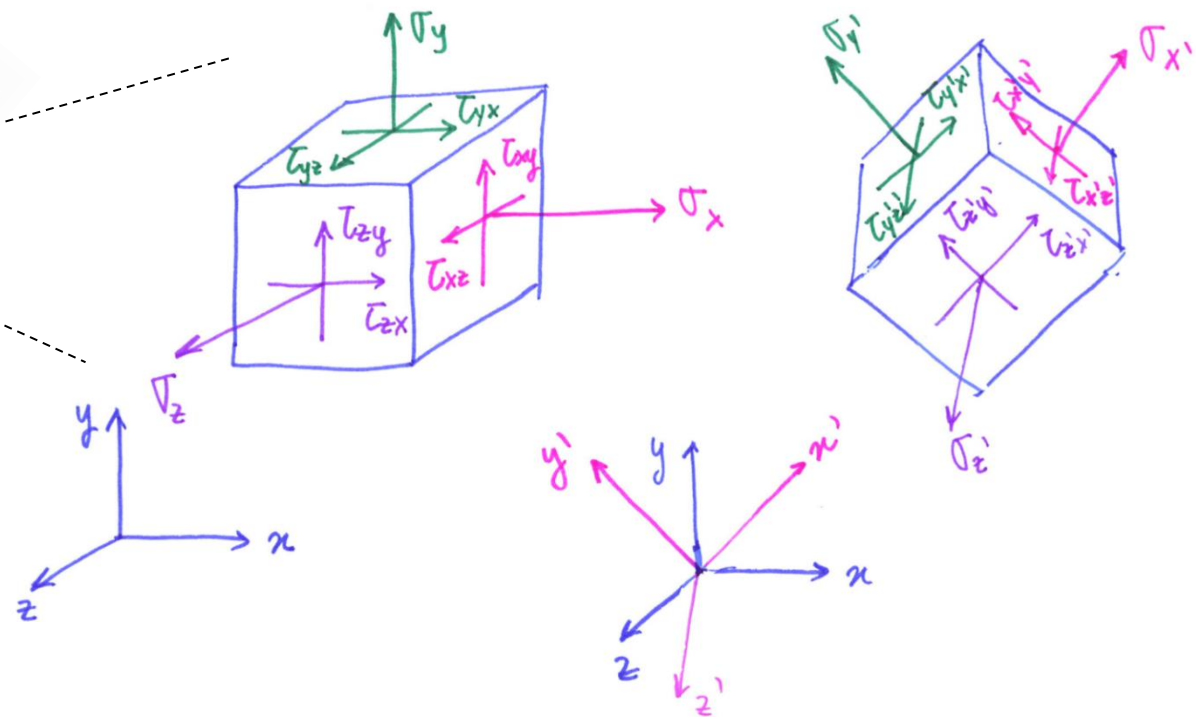
General stress state

The general state of stress at a point is characterized by three independent normal stress components and three independent shear stress components, and is represented by the **stress tensor**. The combination of the state of stress for every point in the domain is called the **stress field**.

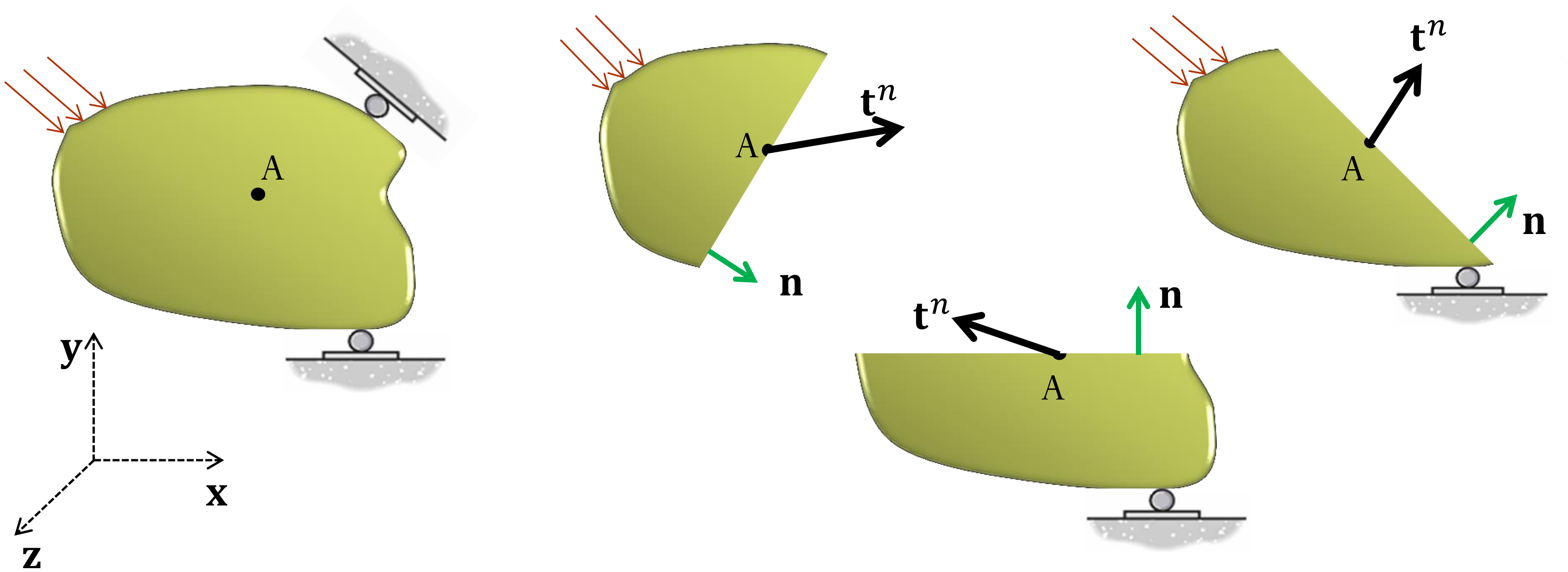




$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Note that the stress is a physical quantity and as such, it is independent of the coordinate system chosen to represent it.

For any surface that divides the body (imaginary or real surface), the action of one part of the body on the other is equivalent to the system of distributed internal forces and moments and it is represented by the stress vector $\mathbf{t}^n$ (also called traction), defined on the surface with normal unit vector **n.**



The state of stress at a point in the body is defined by all the stress vectors $\mathbf{t}^n$associated with all planes (infinite in number) that pass through that point.

Cauchy's stress theorem states that there exists a stress tensor **T** (which is independent of **n**), such that $\mathbf{t}^n$ is a linear function of **n**:

$$\mathbf{t}^n = \mathbf{T}\,\mathbf{n} \qquad \boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Stress components

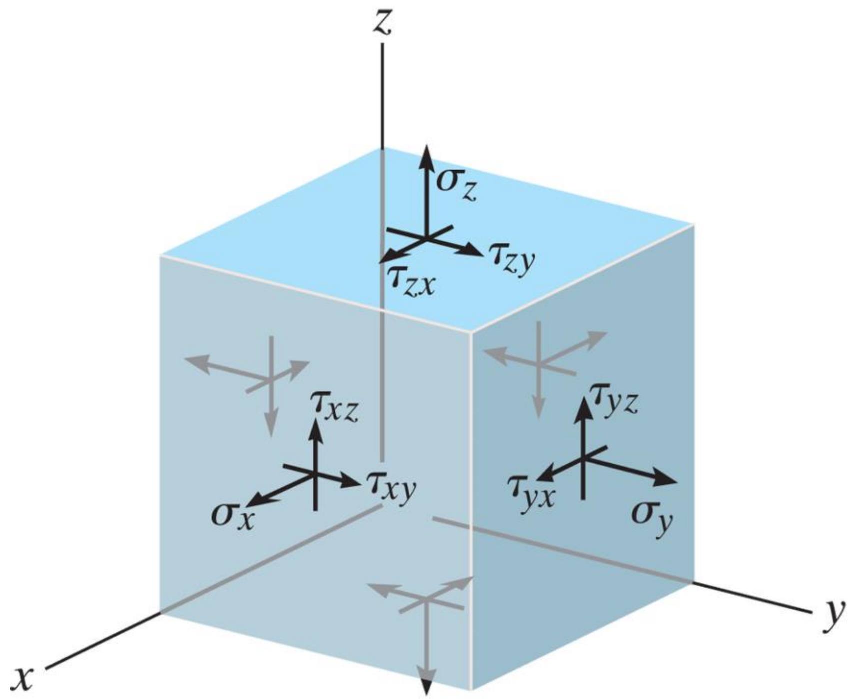
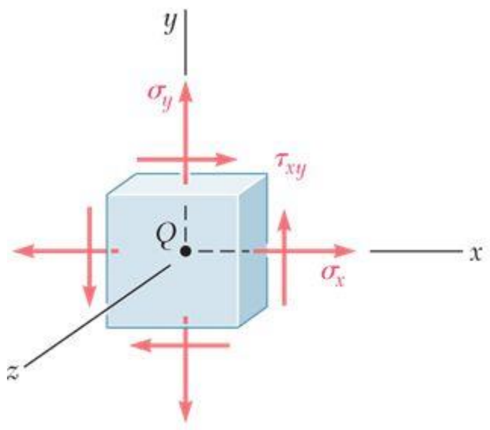


Figure: 01_11

- Sign convention:
  - Positive normal stress acts outward from all faces
  - Positive shear stress points towards the positive axis direction in a positive face
  - Positive shear stress points towards the negative axis direction in a negative face

Plane Stress

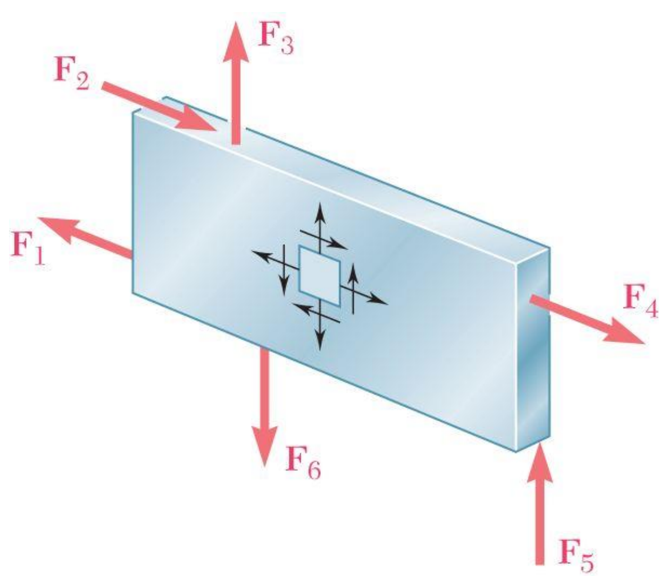


- Two faces of the cube element are stress free

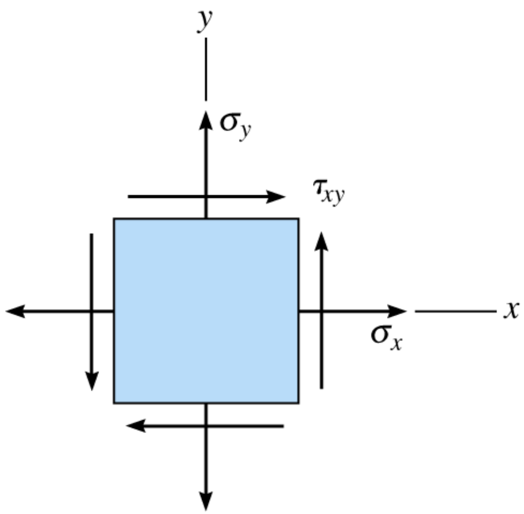
$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

- Example:

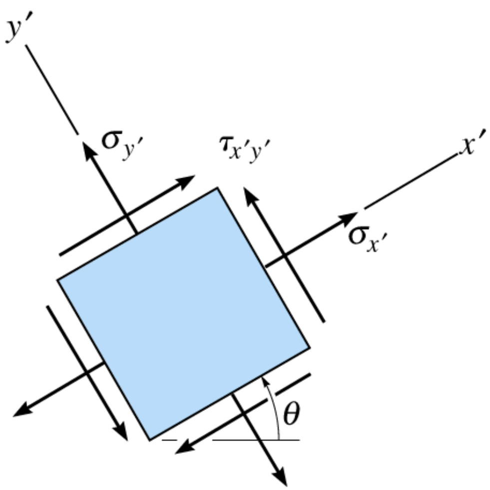
Thin plates subject to forces acting in the mid-plane of the plate



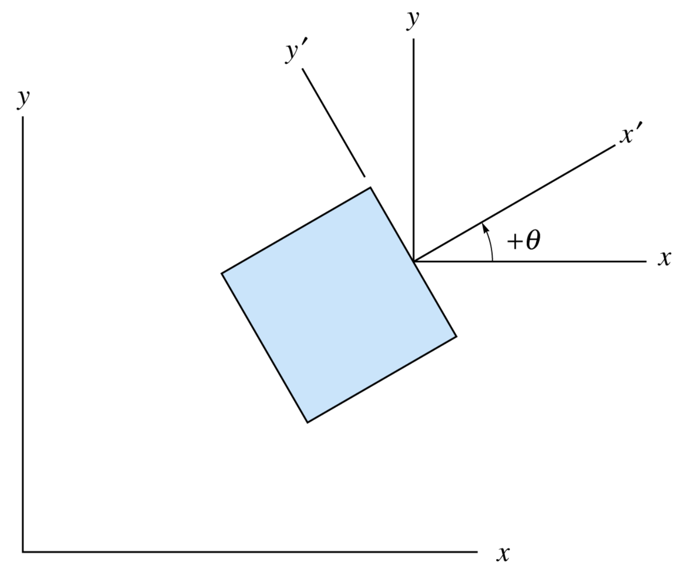
Plane Stress Transformation



(a)



(b)



- Sign convention:
  - Both the x-y and x'-y' system follow the right-hand rule
  - The orientation of an inclined plane (on which the normal and shear stress components are to be determined) will be defined using the angle θ. The angle θ is measured from the positive x to the positive x'-axis. It is positive if it follows the curl of the right-hand fingers.

For two-dimensional problems:

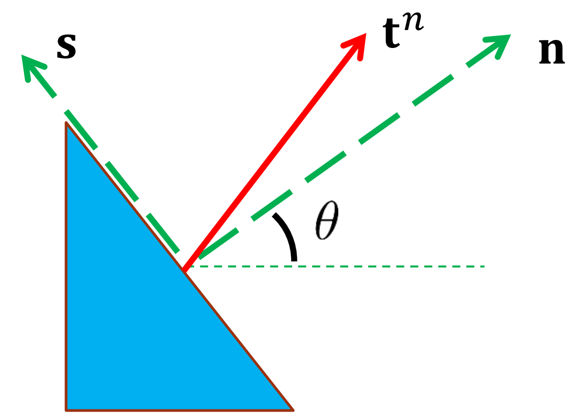
$$\mathbf{n} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\mathbf{t}^n = \mathbf{T}\,\mathbf{n}$$

$$\mathbf{s} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$$\mathbf{t}^n = \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \begin{bmatrix} \sigma_x \cos(\theta) + \tau_{xy} \sin(\theta) \\ \sigma_y \sin(\theta) + \tau_{xy} \cos(\theta) \end{bmatrix}$$

- The normal component of the traction is given by



$$\begin{aligned} \sigma_n &= \mathbf{n} \cdot \mathbf{t}^n = \mathbf{n} \cdot \mathbf{T}\,\mathbf{n} \\ &= \sigma_x \cos^2(\theta) + 2\,\tau_{xy} \sin(\theta) + \sigma_y \sin^2(\theta) \end{aligned}$$

and is called <u>normal stress</u>

- The tangential component of the traction is given by

$$\begin{aligned} \tau_{ns} &= \mathbf{s} \cdot \mathbf{t}^n = \mathbf{s} \cdot \mathbf{T}\,\mathbf{n} \\ &= (\sigma_y - \sigma_x) \sin(\theta) \cos(\theta) + \tau_{xy} (\cos^2(\theta) - \sin^2(\theta)) \end{aligned}$$

and is called <u>shear stress</u>

- Similarly, we can obtain the normal stress

$$\sigma_s = \mathbf{s} \cdot \mathbf{T}\,\mathbf{s}$$

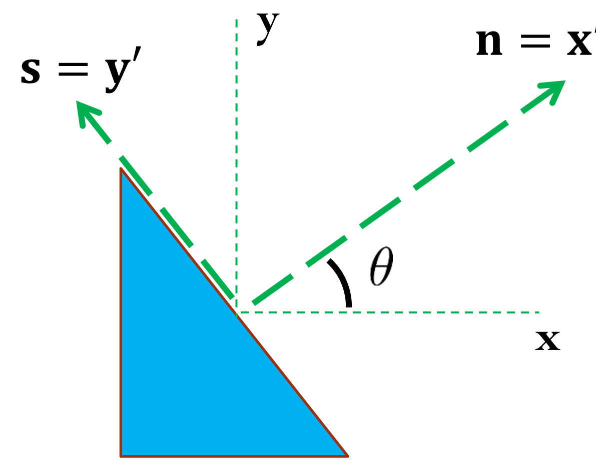
We use the following trigonometric relations…

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \sin(2\theta) = 2\sin\theta\,\cos\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

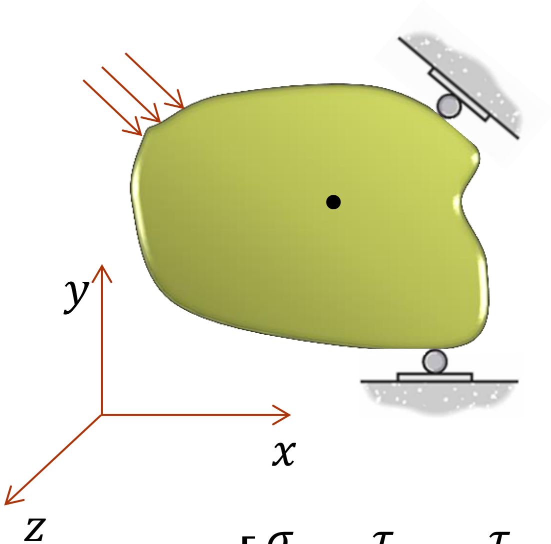
… to get

$$\boxed{\begin{aligned}
\sigma_{x'} &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)\\
\tau_{x'y'} &= -\frac{\sigma_x-\sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)\\
\sigma_{y'} &= \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)
\end{aligned}}$$



Note that: $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$

Principal Stresses



$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

- The stress tensor is a physical quantity and therefore independent of the coordinate system. There are certain invariants associated with every tensor which are also independent of the coordinate system.
  - ✓ First-order tensors (vectors): **magnitude** is the invariant of a vector, since it is independent of the coordinate system chosen to represent the vector.
  - ✓ Second-order tensors (matrices): three independent invariant quantities associated with it. One set of such invariants are the **eigenvalues** of the stress tensor, which are called the principal stresses. The **eigenvectors** define the principal direction vectors.

- Because of symmetry, the stress tensor $\boldsymbol{T}$ has real eigenvalues λ and mutually perpendicular eigenvectors $\boldsymbol{v}$ such that

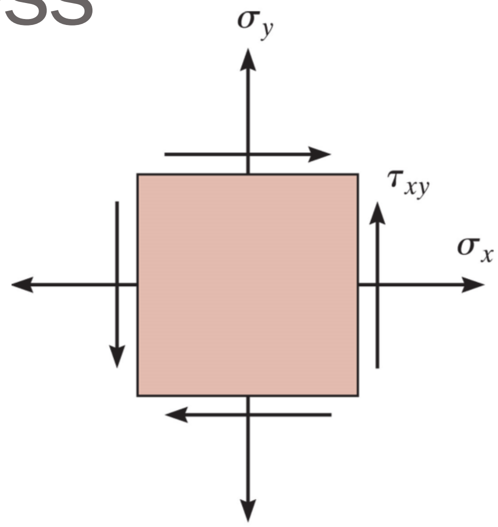
$$\mathbf{T}\,\mathbf{v} = \lambda\,\mathbf{v} \Longrightarrow (\mathbf{T} - \lambda\mathbf{I})\,\mathbf{v} = 0$$

- From linear algebra we know that a system of linear equations $\mathbf{A}\ \mathbf{v} = \mathbf{0}$ has a non-zero solution $\mathbf{v}$ if, and only if, the determinant of the matrix $\mathbf{A}$ is zero, that is

$$\det(\mathbf{T} - \lambda\,\mathbf{I}) = 0$$

Principal Stresses in plane stress

For two-dimensional problems: $\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$



$$\det(\mathbf{T} - \lambda\,\mathbf{I}) = 0$$

$$\det \begin{bmatrix} \sigma_x - \lambda & \tau_{xy} \\ \tau_{xy} & \sigma_y - \lambda \end{bmatrix} = 0$$

$$\begin{aligned} (\sigma_x - \lambda)(\sigma_y - \lambda) - \tau_{xy}^2 &= 0 \\ \lambda^2 - (\sigma_x + \sigma_y)\,\lambda + \sigma_x\,\sigma_y - \tau_{xy}^2 &= 0 \end{aligned}$$

$$\lambda = \frac{(\sigma_x + \sigma_y) \pm \sqrt{((\sigma_x + \sigma_y)^2 - 4\,\sigma_x\,\sigma_y + 4\,\tau_{xy}^2)}}{2}$$

$$\lambda = \frac{(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x^2 + \sigma_y^2 + 2\,\sigma_x\,\sigma_y - 4\,\sigma_x\,\sigma_y + 4\,\tau_{xy}^2)}}{2}$$

$$\lambda = \frac{(\sigma_x + \sigma_y)}{2} \pm \sqrt{\left(\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right)}$$

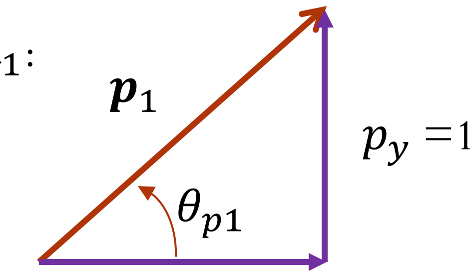
$$\lambda = \sigma_{ave} \pm R$$

$$\boxed{\begin{aligned} \sigma_1 &= \sigma_{ave} + R \\ \sigma_2 &= \sigma_{ave} - R \end{aligned}}$$

Principal Directions

$$(\mathbf{T} - \lambda \mathbf{I})\,\mathbf{p} = 0$$

Obtaining the eigenvector $\boldsymbol{p}_1 = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ corresponding to the first eigenvalue λ_1:

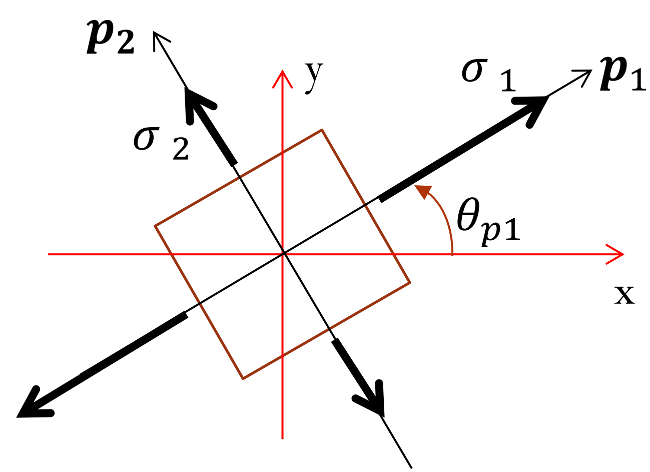


$$p_x = \frac{\tau_{xy}}{\lambda_1 - \sigma_x}$$

$$\begin{bmatrix} \sigma_x - \lambda_1 & \tau_{xy} \\ \tau_{xy} & \sigma_y - \lambda_1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = 0$$

$$(\sigma_x - \lambda_1) p_x + \tau_{xy}\, p_y = 0 \;\Rightarrow\; p_x = \frac{-\tau_{xy}}{\sigma_x - \lambda_1} p_y = \frac{\tau_{xy}}{\lambda_1 - \sigma_x} p_y$$

$$\boxed{\begin{aligned} \theta_{p1} &= \tan^{-1}\left(\frac{v_2}{v_1}\right) = \tan^{-1}\left(\frac{\lambda_1 - \sigma_x}{\tau_{xy}}\right) \\ \theta_{p2} &= \theta_{p1} + 90^o \end{aligned}}$$



The principal stresses represent the maximum and minimum normal stress at the point.

When the state of stress is represented by the principal stresses, no shear stress will act on the element (eigenvectors are orthogonal vectors)

$$\mathbf{T}\,\mathbf{p_1} = \lambda\,\mathbf{p_1} \;\Rightarrow\; \mathbf{p_2} \cdot \mathbf{T}\,\mathbf{p_1} = \lambda\,\mathbf{p_2} \cdot \mathbf{p_1} = 0 \;\Rightarrow\; \tau_{12} = 0$$