Chapter 12: Deflection of Beams and Shafts

Chapter Objectives

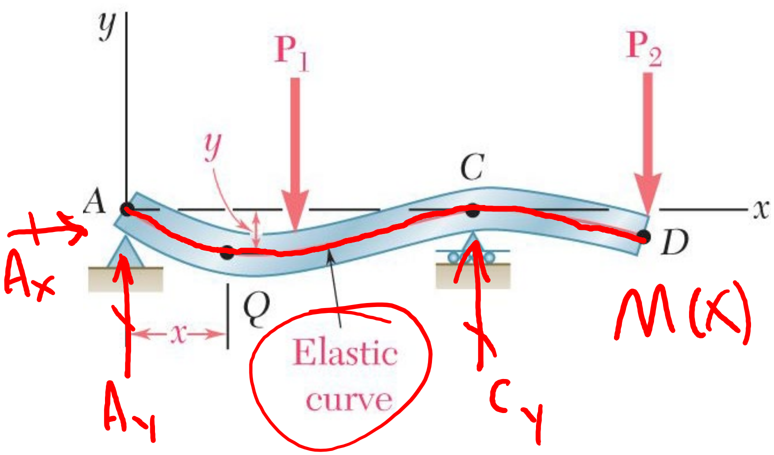
✓ Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:

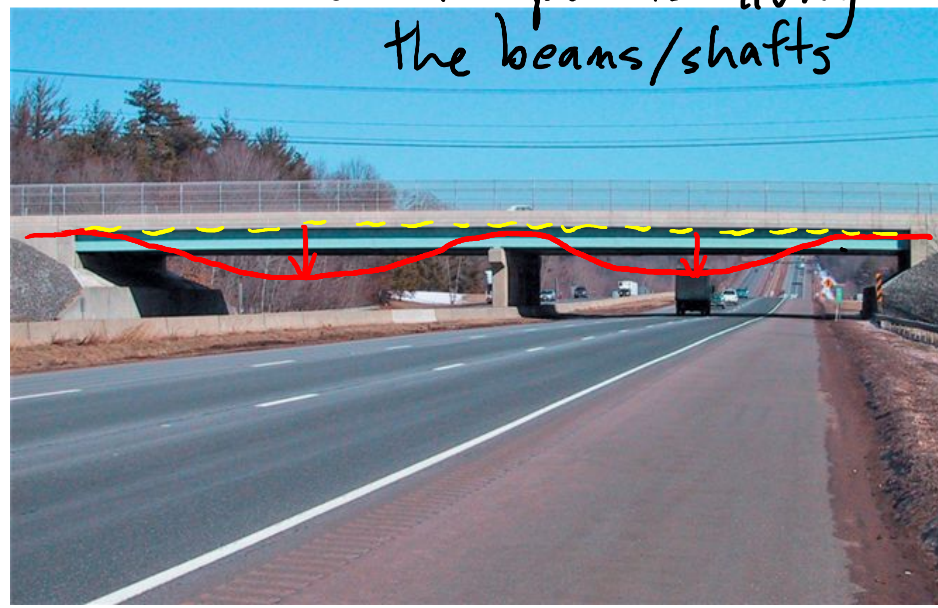
- ➢The integration method — solve a governing O.D.E.
- ➢The method of superposition

Deflection of beams

- **Goal**: Determine the deflection and slope at specified points of beams and shafts

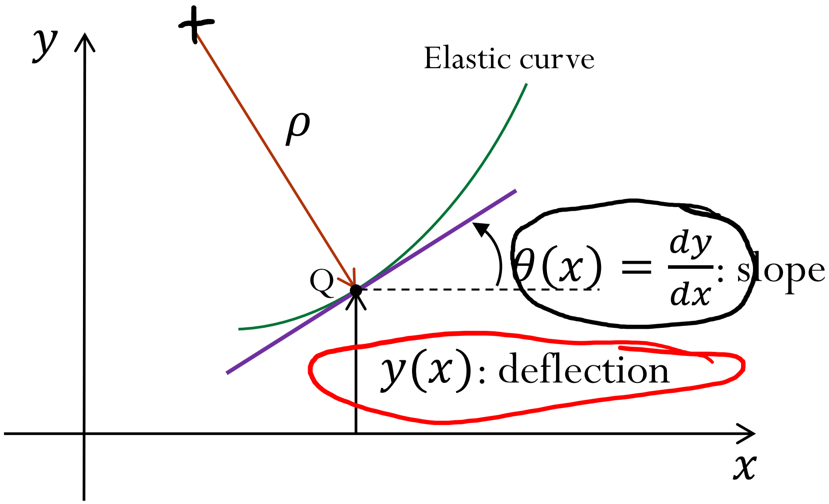
  of all points along the beams/shafts





- **Solve statically indeterminate beams**: where the number of reactions at the supports exceeds the number of equilibrium equations available.
- **Maximum deflection of the beam:** Design specifications of a beam will generally include a maximum allowable value for its deflection

Moment-Curvature equation:



$$\frac{1}{\rho} = \frac{M(x)}{E \cdot I}$$

flexural rigidity

radius of curvature

From calculus: $k = \frac{y''}{(1+y'^2)^{3/2}}$

$y' = \frac{dy}{dx} = \theta(x) = \text{slope}$

Assume $\theta(x)$ is small

$\Rightarrow k = \frac{y''}{(1+0^2)^{3/2}} \approx y''(x)$

$\Rightarrow k = \frac{1}{\rho} \Rightarrow y'' = \frac{d^2y}{dx^2} = \frac{M(x)}{E \cdot I}$

Solve $y'' = \frac{M(x)}{EI}$ to find $y(x)$, which describes the deformed N.A. of the beam.

$$\frac{d^2y}{dx^2} = \frac{M(x)}{E\,I}$$

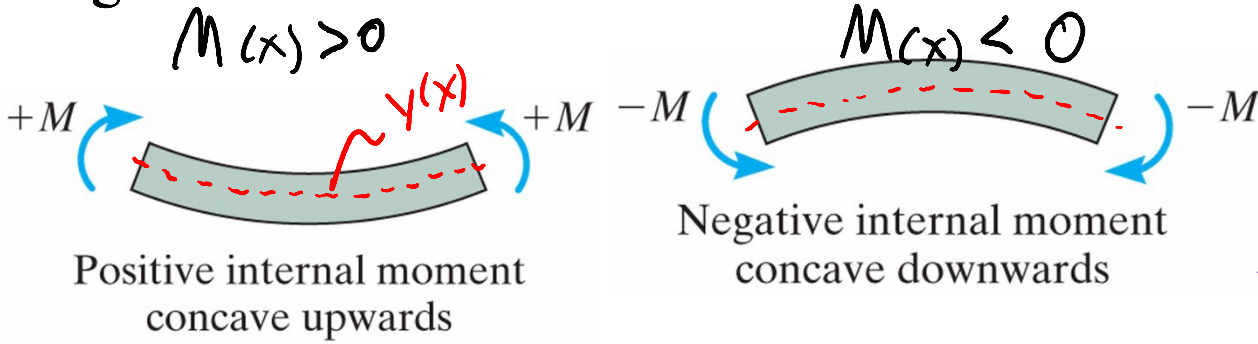
Governing equation of the elastic curve

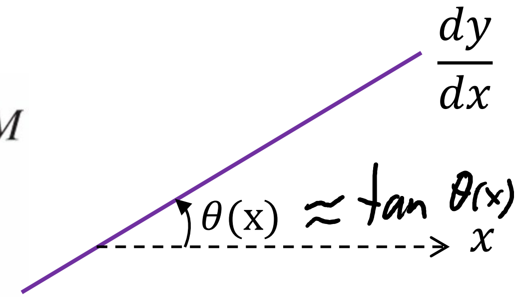
- Elastic curve equation for constant E and I: $E\,I\,y'' = M(x)$
- Differentiating both sides gives: $E\,I\,y''' = \frac{dM(x)}{dx} = V(x)$ Transverse Shear Force
- Differentiating again: $E\,I\,y'''' = \frac{dV(x)}{dx} = -w(x)$ {distributed load
  $E I \cdot y^{IV}$
- In summary, we have:

$y(x)$: deflection curve
$y'(x)$: slope $= \theta(x)$ → $\tan\theta \approx \theta$ for small θ (in radians)
$E\,I\,y''(x)$: bending moment $M(x)$
$E\,I\,y'''(x)$: shear force $V(x)$
$E\,I\,y''''(x)$: distributed load $-w(x)$

- **Sign conventions:**

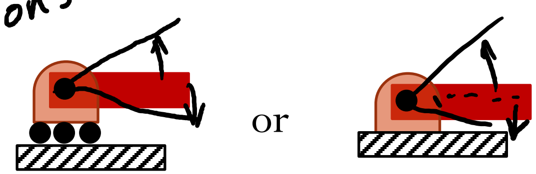
$EI\,y'' = M(x)$





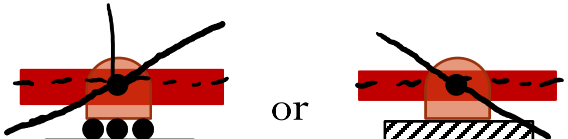
- **Boundary conditions**

θ(x) not constrained



Roller: $y = 0$; y' not specified

Pin: $y = 0$



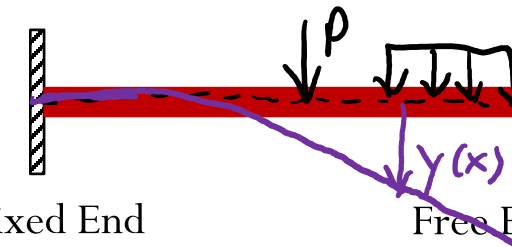
Roller: $y = 0$

$\theta(x^-) = \theta(x^+)$

Pin: $y'(x^-) = y'(x^+)$

slope is continuous

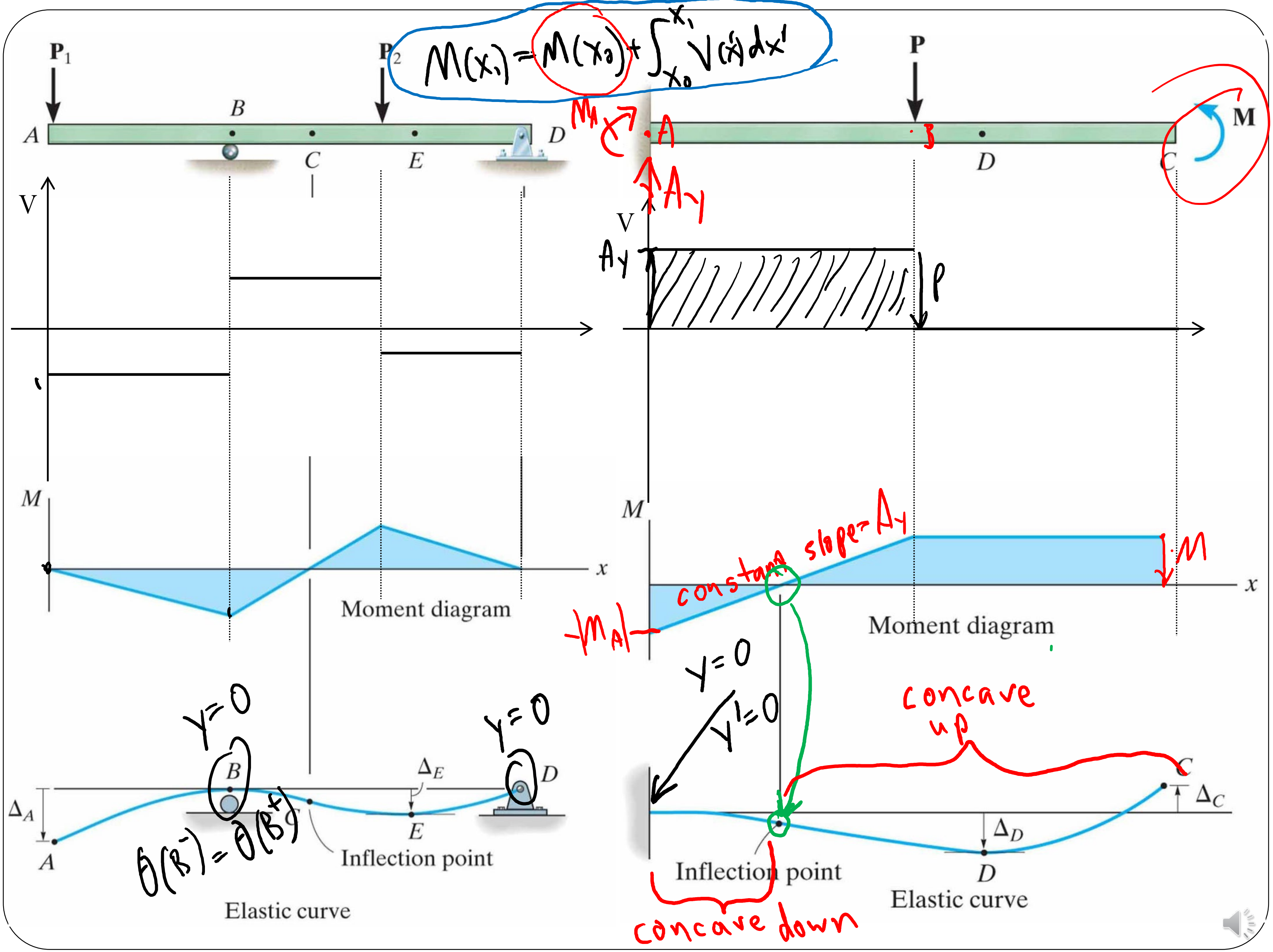
Cantilever



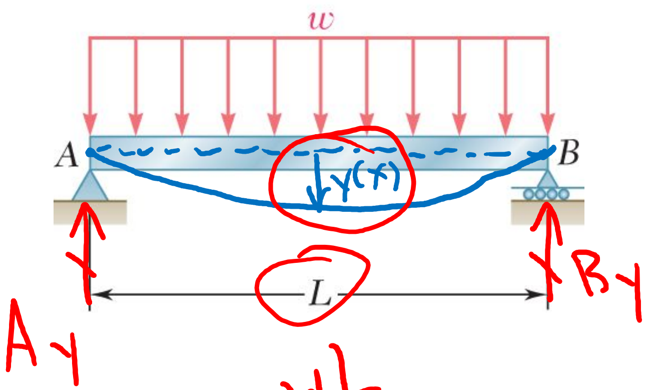
Fixed End: $y = 0$

$y' = \theta = 0$

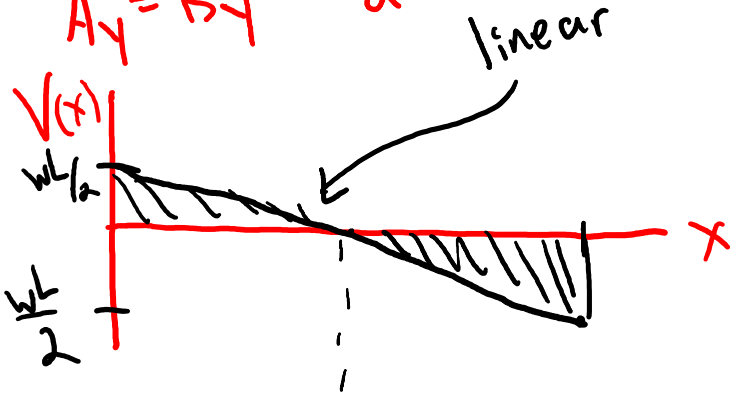
Free End: No constraint of the deflection or the slope



$M(x_1) = M(x_0) + \int_{x_0}^{x_1} V(x')\,dx'$
P_1
P_2
A
B
C
D
E
V
M
x
Moment diagram
Y=0
Y=0
$\theta(B^-) = \theta(B^+)$
Δ_A
Δ_E
Inflection point
Elastic curve
P
M
M_A
A_y
B
C
D
P
constant slope = A_y
$-|M_A|$
Y=0
Y'=0
concave up
concave down
Δ_C
Δ_D
Inflection point
Elastic curve



$A_y = B_y = \frac{wL}{2}$



$[w] = \frac{\text{force}}{\text{length}}$

$[L^4] = \text{length}^4$

$[EI] = \text{force} \times \text{length}^2$

$\Rightarrow \left[\frac{wL^4}{EI}\right] = \text{length}$

$[y(x)] = \text{length}$

$\left[\frac{wL^4}{EI}\right]$

$$y'' = \frac{M(x)}{EI}$$

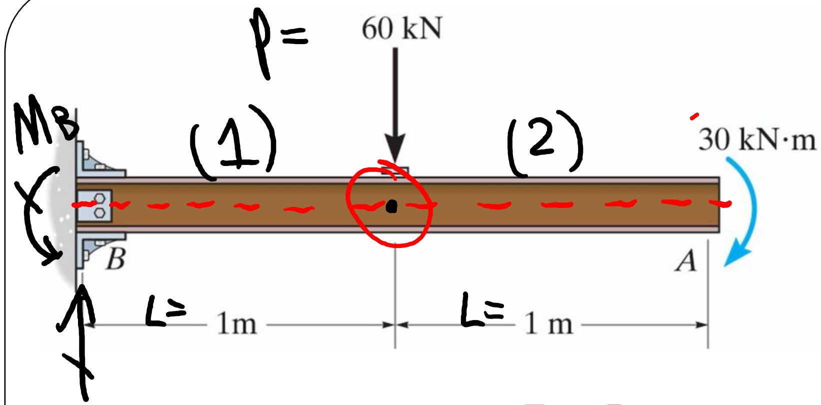
$$y(L) = 0 = \frac{wL^2}{2EI}\underbrace{\left(\frac{L^2}{6} - \frac{L^2}{12}\right)}_{L^2/12} + C_1 \cdot L$$

$$0 = \frac{wL^2}{2EI} \cdot \frac{L^2}{12} + C_1 \cdot L \Rightarrow C_1 = -\frac{wL^3}{24EI}$$

$$y(x) = \frac{wL^2}{2EI}\left(\frac{x^3}{6L} - \frac{x^4}{12L^2}\right) - \frac{wL^3 x}{24EI}$$

$$= \frac{wL^2}{EI}\left(-\frac{x^4}{24L^2} + \frac{x^3}{12L} - \frac{x \cdot L}{24}\right)\frac{L^2}{L^2}$$

$$= \frac{wL^4}{EI}\underbrace{\left[-\frac{1}{24}\left(\frac{x}{L}\right)^4 + \frac{1}{12}\left(\frac{x}{L}\right)^3 - \frac{1}{24}\left(\frac{x}{L}\right)\right]}_{\text{nondimensional}}$$



B_y

$$y_1(x) = \frac{P \cdot x^3}{6EI} - \frac{(P \cdot L + M)x^2}{2EI}$$

$$y_1'(x) = \frac{Px^2}{2EI} - \frac{(PL+M) \cdot x}{EI}$$

$$y_1(L) = \frac{PL^3}{6EI} - \frac{(P \cdot L + M)L^2}{EI}$$

$$y_1'(L) = \frac{PL^2}{2EI} - \frac{(P \cdot L + M) \cdot L}{EI}$$

$$M(x) = \begin{cases} P \cdot x - (P \cdot L + M), & 0 < x < L \\ -M, & L < x < 2L \end{cases}$$

$$y_2'(x) = -\frac{PL^2}{2EI} - \frac{M \cdot x}{EI}$$

$$y'' = \frac{M(x)}{EI}$$

$$y_1(L) = y_2(L)$$

$$y_1'(L) = y_2'(L)$$

$$y_2(x) = y_2(L) + \int_L^x y_2'(x') \cdot dx'$$

$$= \frac{PL^3}{6EI} - \frac{(PL+M)L^2}{EI} + \int_L^x \left(-\frac{PL^2}{2EI} - \frac{M \cdot x'}{EI}\right) dx'$$

$$= \frac{PL^3}{6EI} - \frac{(PL+M)L^2}{EI} - \frac{PL^2(x-L)}{2EI} - \frac{M}{EI}\int_L^x x' dx'$$

$$\frac{1}{2}(x^2 - L^2)$$

$$= \frac{PL^3}{6EI} - \frac{PL^3 + ML^2}{EI} - \frac{PL^2 x}{2EI} + \frac{PL^3}{2EI} - \frac{M}{2EI}(x^2 - L^2)$$

$$= -\frac{PL^3}{3EI} - \frac{PL^2 x}{2EI} + \frac{ML^2}{EI} + \frac{ML^2}{2EI} - \frac{Mx^2}{2EI}$$

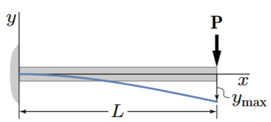
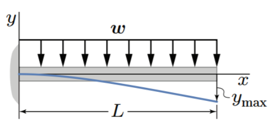
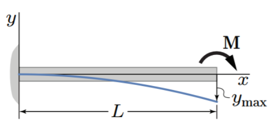
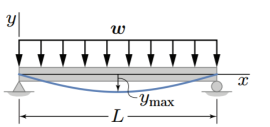
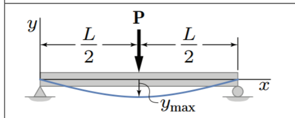
$$y_2(x) = \frac{PL^2}{EI}\left(-\frac{L}{3} - \frac{x}{2}\right) + \frac{M}{2EI}(3L^2 - x^2)$$

Superposition principle

Many common beam deflection solutions have been worked out – see your formula sheet!

If we'd like to find the solution for a loading situation that is not given in the table, we can use superposition to get the answer: represent the load of interest as a combination of two or more loads that are given in the table, and the resulting deflection curve for this loading is simply the sum of each curve from each loading treated separately

Beam Deflection

Diagram	Max. Deflection	Slope at End	Elastic Curve
	y_{max}	θ	$y(x)$
	$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y(x) = \frac{P}{6EI}(x^3 - 3Lx^2)$
	$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y(x) = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
	$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y(x) = -\frac{M}{2EI}x^2$
	$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y(x) = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
	$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $0 \le x \le \frac{L}{2}$ $y(x) = \frac{P}{48EI}(4x^3 - 3L^2x)$

Obtain the deflection at point A using the superposition method – compare with the result obtained using the integration method!

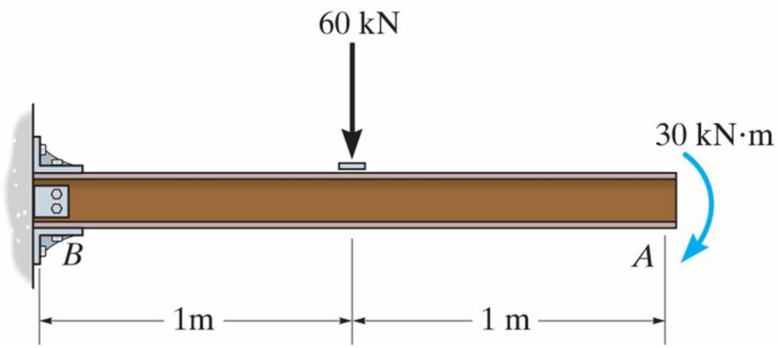
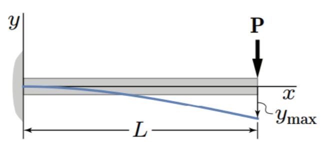
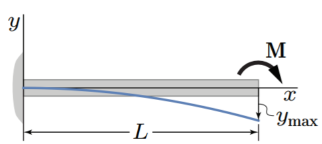
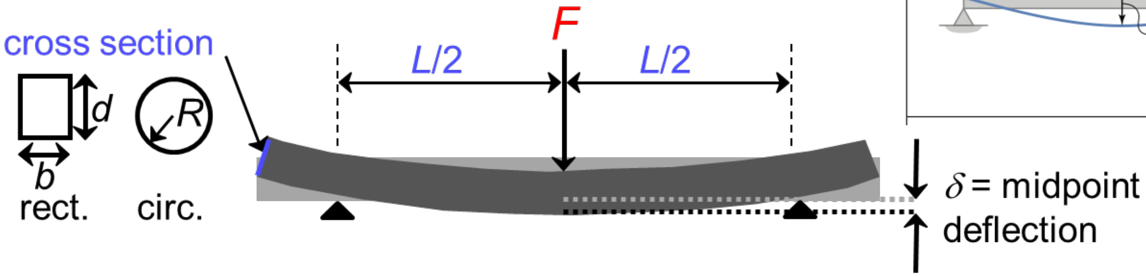


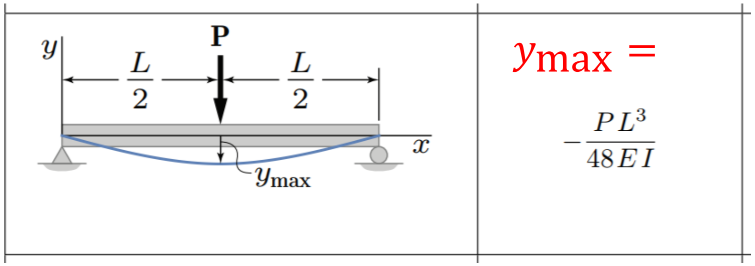
Diagram	Max. Deflection	Slope at End	Elastic Curve
	y_{max}	θ	$y(x)$
	$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y(x) = \frac{P}{6EI}\left(x^3 - 3Lx^2\right)$
	$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y(x) = -\frac{M}{2EI}x^2$

Practical application: measure stiffness of brittle material

It can be difficult to perform a tension test on brittle materials – can easily crack at grips and can only withstand small amount of strain

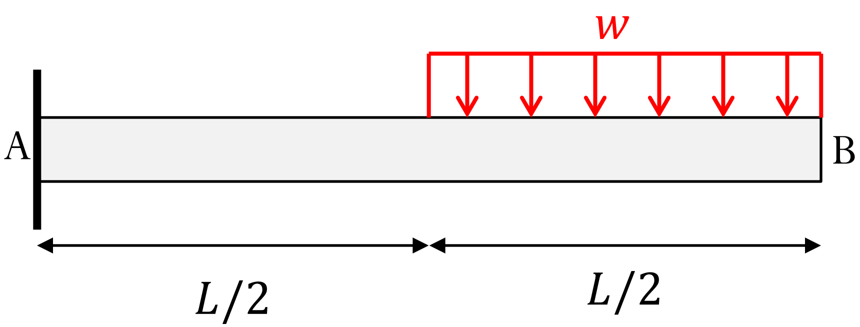
Instead, a 3 point bending test is often used



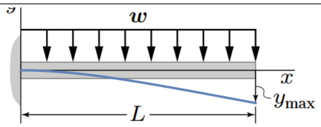


Find an expression for the stiffness E of the material, given the geometry, applied load F, and deflection δ at the midpoint

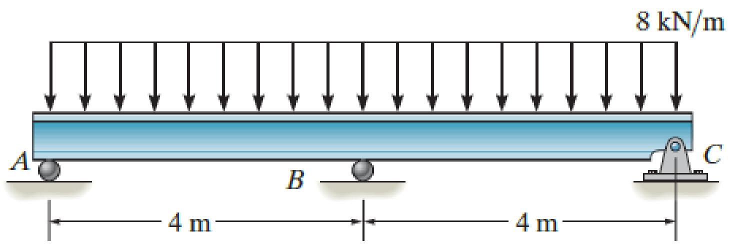
Assuming failure occurs at a force F_f, find an expression for the stress at failure σ_f

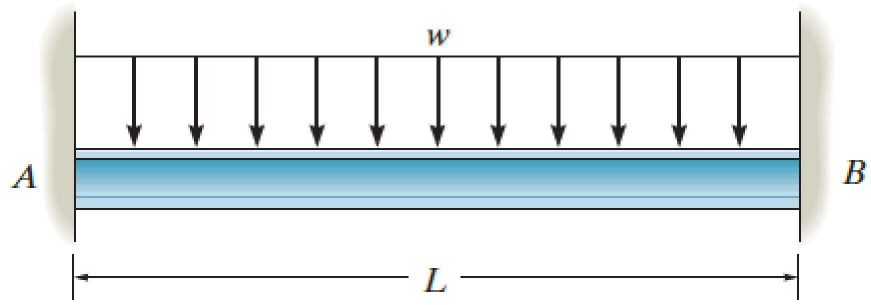


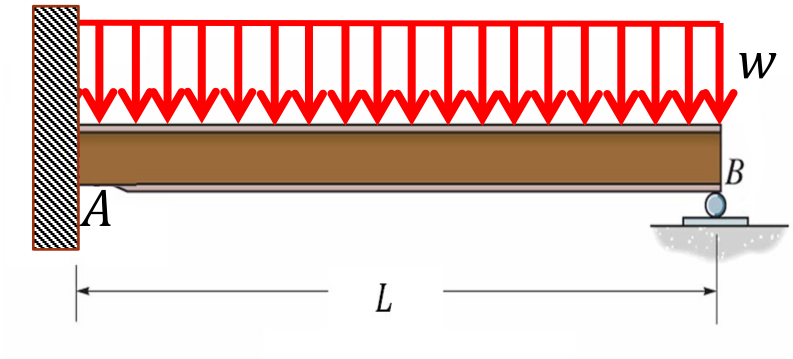
Find the deflection at point B at the end of the cantilever beam

Beam Deflection			
Diagram	Max. Deflection	Slope at End	Elastic Curve
	y_{max}	θ	$y(x)$
	$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y(x) = -\frac{w}{24EI}\left(x^4 - 4Lx^3 + 6L^2x^2\right)$

Indeterminate problems



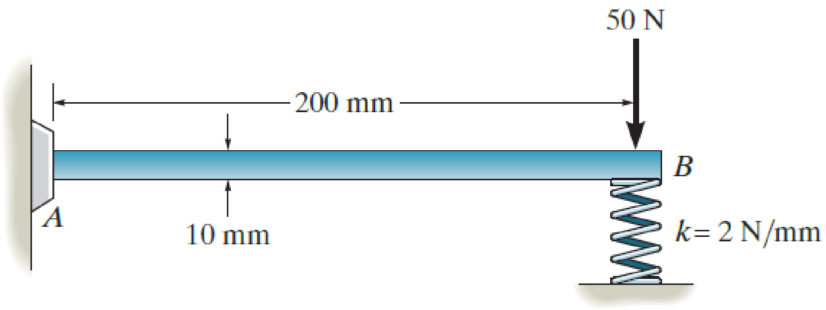


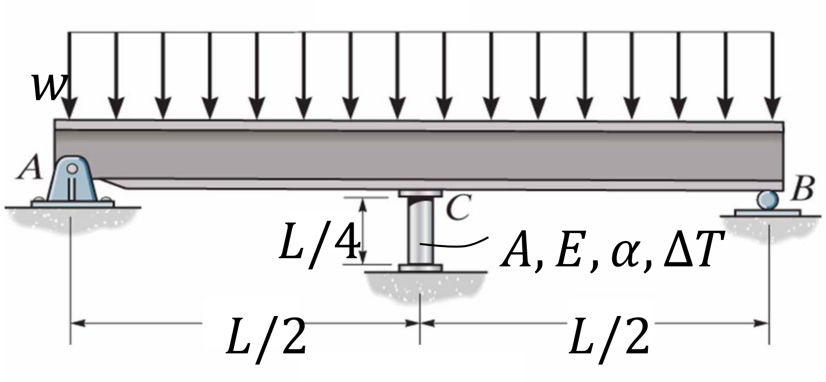


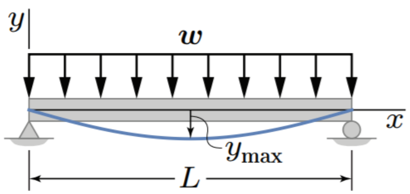
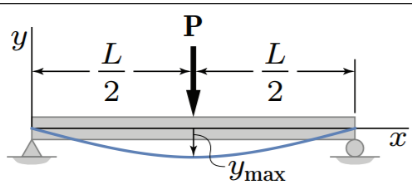
Obtain the reaction at the support B using

- Integration method
- Superposition method

Determine the deflection at B

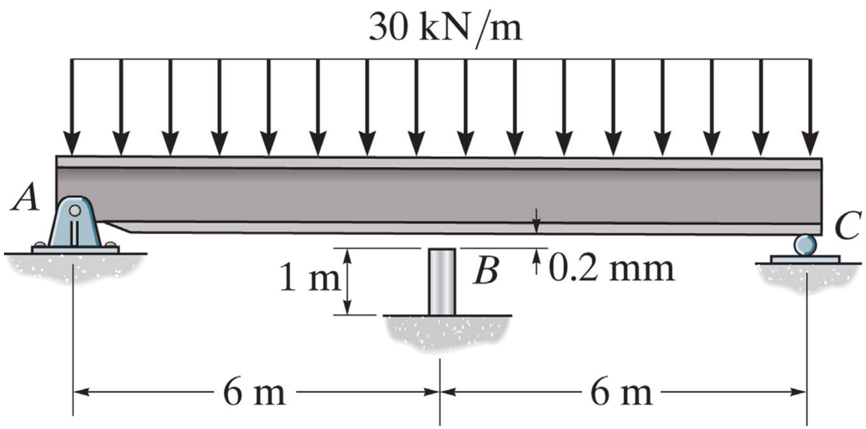


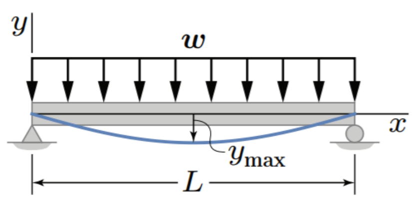
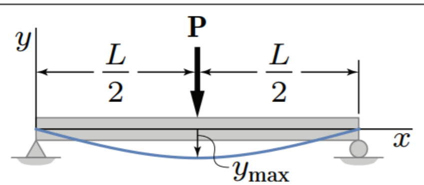


	$y$max $-\dfrac{5wL^4}{384EI}$
	$y$max $-\dfrac{PL^3}{48EI}$

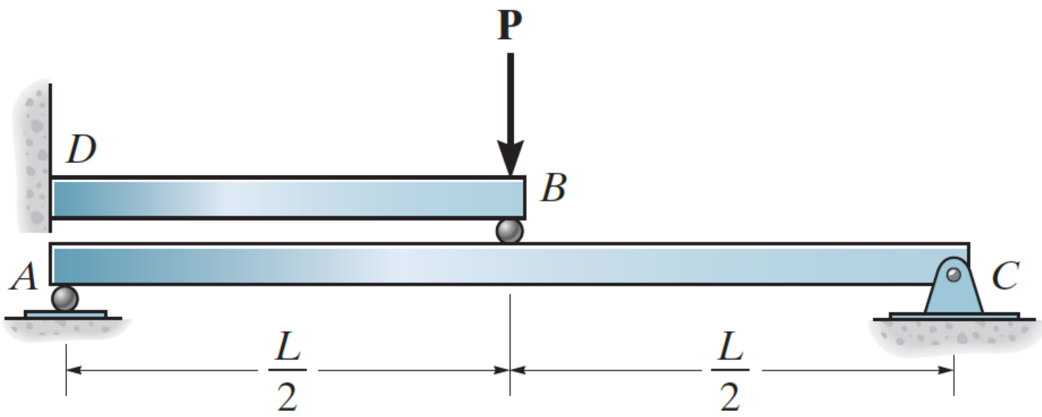
The beam is supported by a pin at A, a roller at B, and a deformable post at C. The post has length L/4, cross-sectional area A, modulus of elasticity E, and thermal expansion coefficient α. The beam has constant moment of inertia $\text{I} = A\,L^2$ and modulus of elasticity E (the same as the post).

(a) Determine the internal force in the post when the distributed load w is applied AND the post experiences an increase in temperature ΔT, where $\Delta T > 0$.

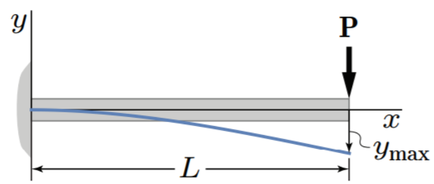
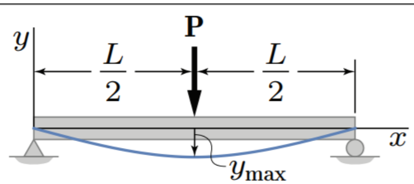


	y_{max} $-\frac{5wL^4}{384EI}$
	y_{max} $-\frac{PL^3}{48EI}$

Before the uniform distributed load is applied on the beam, there is a small gap of 0.2 mm between the beam and the post at B. Determine the support reactions at A, B, and C. The post at B has a diameter of 40 mm, and the moment of inertia of the beam is $I = 875 \times 10^6\ \text{mm}^4$. The post and the beam are made of material having a modulus of elasticity of $E = 200$ GPa.



Find the reaction force at point C using superposition methods

Diagram	Max. Deflection y_{max}
	$-\frac{PL^3}{3EI}$
	$-\frac{PL^3}{48EI}$