TAM251_Chapter12 _DeflectionBeams_prelecture_Johnson

Thursday, July 29, 2021 1:16 PM

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Chapter 12: Deflection of Beams and Shafts

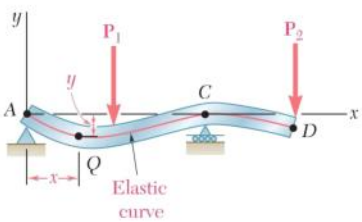
Chapter Objectives

✓ Determine the deflection and slope at specific points on beams and shafts, using various analytical methods including:

- The integration method
- The method of superposition

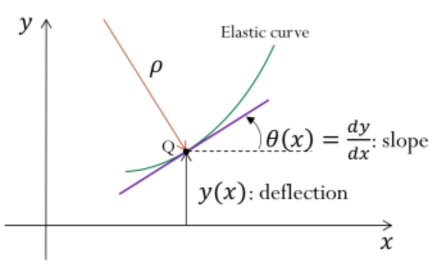
Deflection of beams

- **Goal**: Determine the deflection and slope at specified points of beams and shafts



- **Solve statically indeterminate beams**: where the number of reactions at the supports exceeds the number of equilibrium equations available.
- **Maximum deflection of the beam:** Design specifications of a beam will generally include a maximum allowable value for its deflection

- **Moment-Curvature equation:**



$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

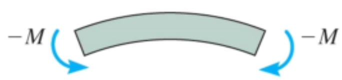
Governing equation of the elastic curve

- Elastic curve equation for constant E and I: $E\,I\,y'' = M(x)$
- Differentiating both sides gives: $E\,I\,y''' = \frac{dM(x)}{dx} = V(x)$
- Differentiating again: $E\,I\,y'''' = \frac{dV(x)}{dx} = -w(x)$
- In summary, we have:

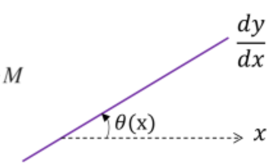
  $y(x)$: deflection
  $y'(x)$: slope
  $E\,I\,y''(x)$: bending moment
  $E\,I\,y'''(x)$: shear force
  $E\,I\,y''''(x)$: distributed load

- **Sign conventions:**

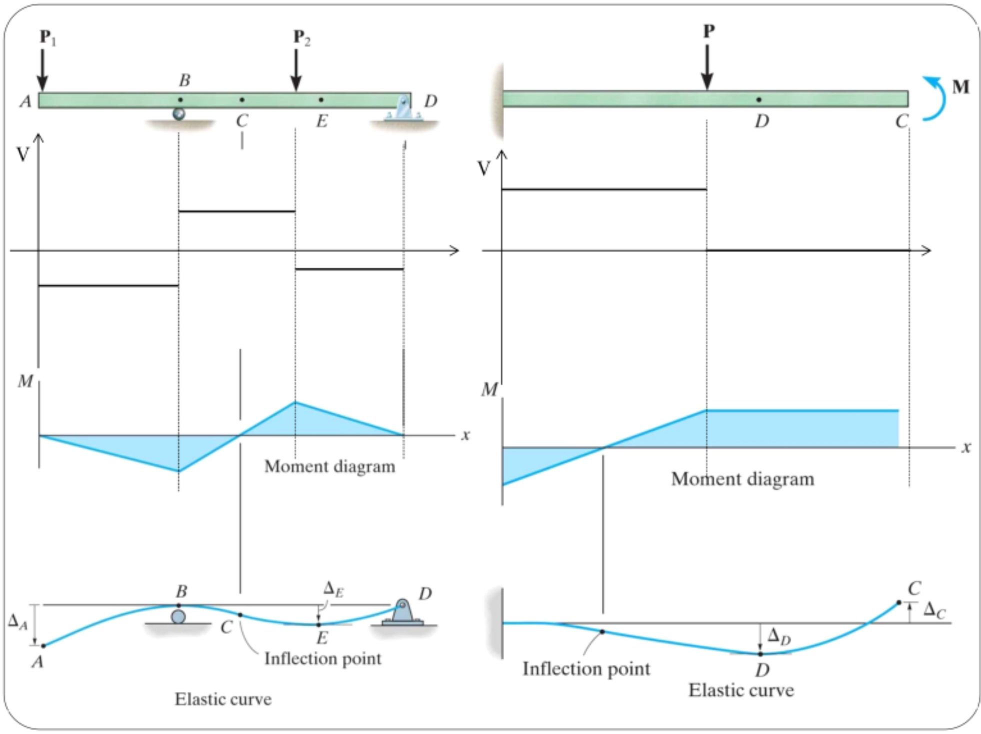
Positive internal moment
concave upwards



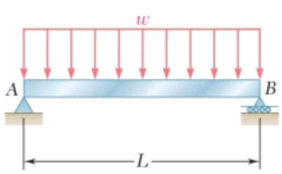
Negative internal moment
concave downwards



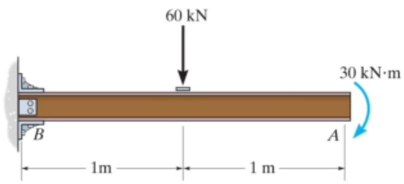
- **Boundary conditions**



P_1
P_2
A
B
C
E
D
V
M
x
Moment diagram
Δ_A
Δ_E
Inflection point
Elastic curve
P
M
D
C
V
M
x
Moment diagram
C
Δ_C
Δ_D
D
Inflection point
Elastic curve



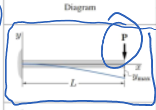
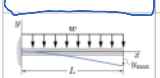
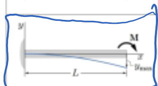
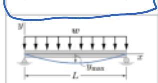
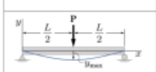
w
A
B
L

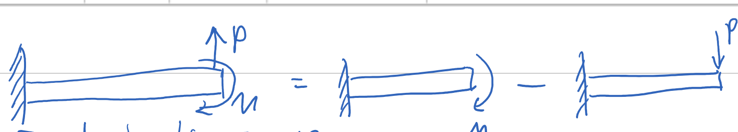


Superposition principle

Many common beam deflection solutions have been worked out – see your formula sheet!

If we'd like to find the solution for a loading situation that is not given in the table, we can use superposition to get the answer: represent the load of interest as a combination of two or more loads that are given in the table, and the resulting deflection curve for this loading is simply the sum of each curve from each loading treated separately

Beam Deflection			
Diagram	Max. Deflection y_{max}	Slope at End θ	Elastic Curve $y(x)$
1 	$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y(x) = \frac{P}{6EI}(x^3 - 3Lx^2)$
	$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y(x) = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 	$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y(x) = -\frac{M}{2EI}x^2$
	$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y(x) = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
	$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $0 \le x \le \frac{L}{2}$ $y(x) = \frac{P}{48EI}(4x^3 - 3L^2x)$



Find elastic curve

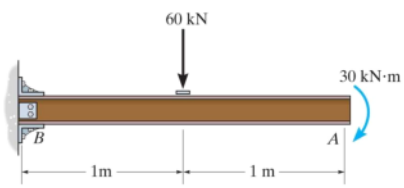
Take 3 − 1

$y(x) = y_3(x) - y_1(x)$

$$y(x) = \frac{-Mx^2}{2EI} - \frac{P}{6EI}\left(x^3 - 3Lx^2\right) \qquad y_{max} = y(L) = \ldots$$

$$\theta(x) = y'(x) = \frac{-Mx}{EI} - \frac{P}{6EI}\left(3x^2 - 6Lx\right) \qquad \theta_{end} = y'(L) = \ldots$$

Obtain the deflection at point A using the superposition method, compare with the result obtained using the integration method!



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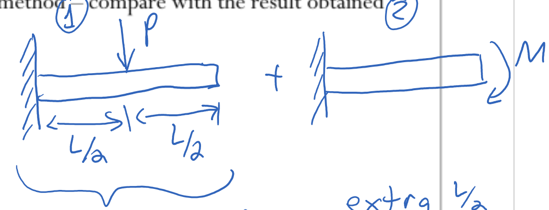
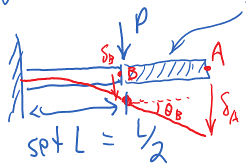
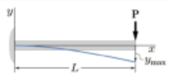
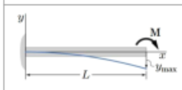




extra L/2 has no load, therefore no curvature

Find slope and deflection at B. $\delta_A = \delta_B + \theta_B \cdot L/2$

	Diagram	Max. Deflection y_{max}	Slope at End θ	Elastic Curve $y(x)$
1		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y(x) = -\frac{P}{6EI}\left(x^3 - 3Lx^2\right)$
2		$\delta_{A_2} = -\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y(x) = -\frac{M}{2EI}x^2$

$$\delta_B = \frac{-P(L/2)^3}{3EI} = \frac{-PL^3}{24EI}$$

$$\theta_B = \frac{-P(L/2)^2}{2EI} = \frac{-PL^2}{8EI}$$

$$(\delta_A)_2 = -\frac{ML^2}{2EI}$$

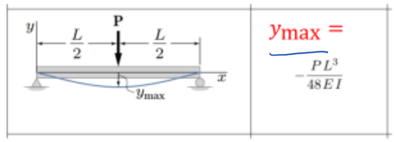
$$(\delta_A)_1 = \frac{-PL^3}{24EI} - \frac{PL^2}{8EI} \cdot \frac{L}{2} = \frac{-PL^3}{EI}\left(\frac{1}{24} + \frac{1}{16}\right) = \frac{-5PL^3}{48EI}$$

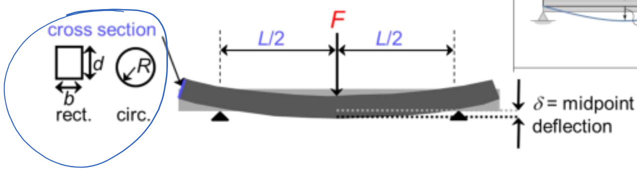
$$\boxed{\delta_A = (\delta_A)_1 + (\delta_A)_2 = -\frac{5PL^3}{48EI} - \frac{ML^2}{2EI}}$$

Practical application: measure stiffness of brittle material

It can be difficult to perform a tension test on brittle materials – can easily crack at grips and can only withstand small amount of strain

Instead, a 3 point bending test is often used





$$y_{max} = y\left(\frac{L}{2}\right) = \delta$$

$$\delta = \frac{FL^3}{48EI}$$

$$\boxed{E = \frac{FL^3}{48 \cdot \delta \cdot I}}$$

$$\sigma = \frac{+M \cdot c}{I}$$

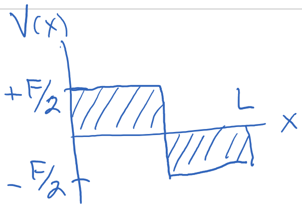
Find an expression for the stiffness E of the material, given the geometry, applied load F, and deflection δ at the midpoint

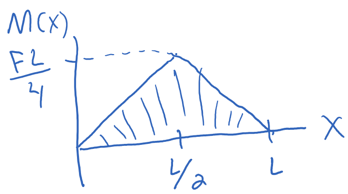
measured by the loading machine

Assuming failure occurs at a force F_f, find an expression for the stress at failure σ_f

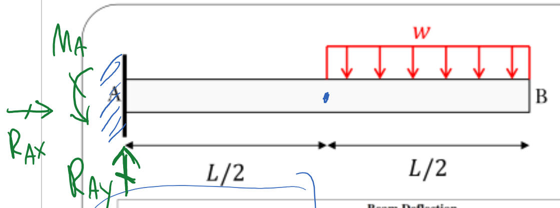
If rectangle, $I = \frac{bd^3}{12}$, $c = d/2$

If circular, $I = \frac{\pi}{4} R^4$, $c = R$

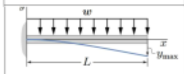


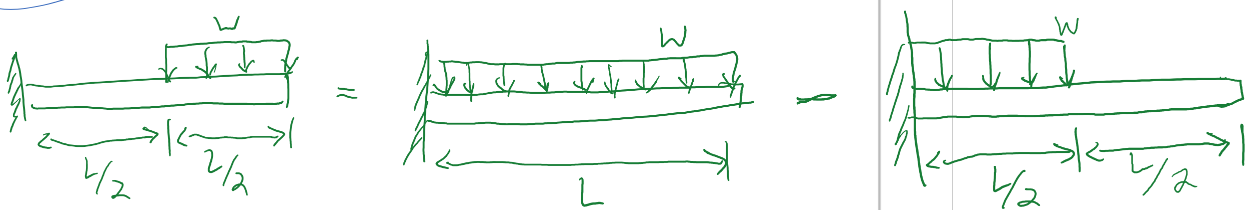


$$\boxed{\sigma_f} = \frac{+\left(\frac{FL}{4}\right)\left(\frac{d}{2}\right)}{\frac{bd^3}{12}} = \boxed{\frac{3 \cdot F \cdot L}{2\, bd^2}}$$



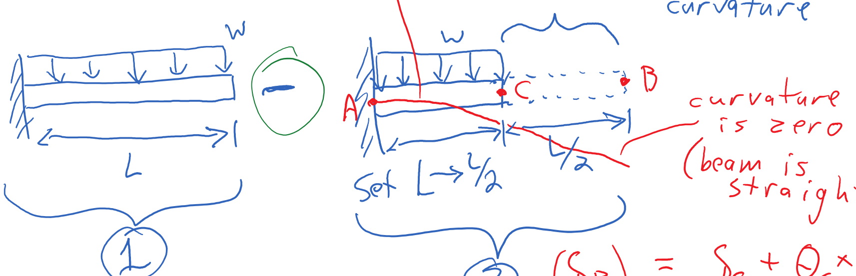
Find the deflection at point B at the end of the cantilever beam

Beam Deflection			
Diagram	Max. Deflection	Slope at End	Elastic Curve
	y_{max}	θ	$y(x)$
	$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y(x) = -\frac{w}{24EI}\left(x^4 - 4Lx^3 + 6L^2x^2\right)$



full sol'n is in the table

can adapt first half to use the table



$(\delta_B)_2 = \delta_c + \theta_c \times \frac{L}{2}$

case 1: Max. deflection is $(\delta_B)_1 = -\frac{wL^4}{8EI}$

set $L \to L/2$

case 2: $\delta_c = \frac{-w(L/2)^4}{8 \cdot EI} = \frac{-wL^4}{128 \cdot EI}$

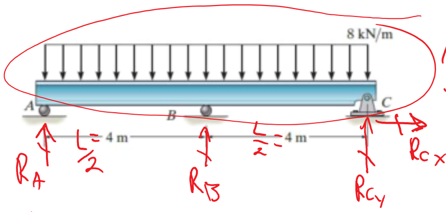
$$\theta_c = \frac{-w(L/2)^3}{6EI} = \frac{-wL^3}{48EI}$$

$$(\delta_B)_2 = \frac{-wL^4}{128EI} - \frac{wL^3}{48EI} \cdot \frac{L}{2} = \frac{-wL^4}{EI}\left(\frac{1}{128} + \frac{1}{96}\right)$$

$$\boxed{\delta_B = (\delta_B)_1 - (\delta_B)_2 = -\frac{wL^4}{EI}\left(\frac{1}{8} - \frac{1}{128} - \frac{1}{96}\right)}$$

Indeterminate problems

More reaction than equilibrium equations.
⇒ Must apply geometric compatibility.



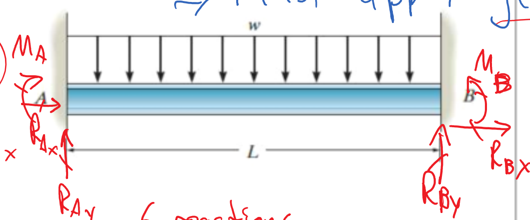
4 reactions, 3 equil. eqns.

$\Sigma F_x = 0$
$\Sigma F_y = 0$
$\Sigma M = 0$

Geometric compatibility

$y(L/2) = 0$ use extra B.C.'s and C.C.'s to set geometric compatibility

Find $y(x)$ in a table, set $y(L/2)=0$ and solve for the coefficients in the $y(x)$ expression

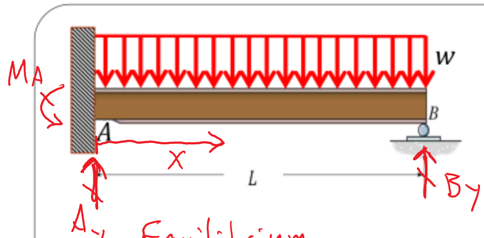


6 reactions
3 equil. eqns. $(R_{Ax} = R_{Bx} = 0)$

$y(0) = 0$
$y'(0) = 0$
$y(L) = 0$
$y'(L) = 0$

Unknowns
4 Reactions $(R_{Ay}, R_{By}, M_A, M_B)$
2 int. constants

Equations
2 equil. eqns $(\Sigma F_y = 0, \Sigma M = 0)$
4 B.C.'s

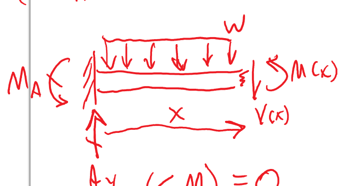


Obtain the reaction at the support B using
- Integration method
- Superposition method

Equilibrium

$\Sigma F_y = 0 \Rightarrow A_y + B_y - w \cdot L = 0 \quad (1)$

$(\Sigma M)_A = 0 \Rightarrow B_y \cdot L + M_A - \frac{wL^2}{2} = 0 \quad (2)$



$(\Sigma M)_x = 0$

$M(x) + w \cdot x \cdot \frac{x}{2} + M_A - A_y \cdot x = 0$

$M(x) = A_y \cdot x - \frac{1}{2} w x^2 - M_A$

$y''(x) = \frac{M(x)}{EI}$ with B.C.'s
$y(0) = 0$
$y'(0) = 0$
$y(L) = 0$ ← geometric compatibility

$y'' = \frac{1}{EI}\left(A_y x - \frac{1}{2} w x^2 - M_A\right)$

$y'(x) = \frac{1}{EI}\left(\frac{1}{2} A_y x^2 - \frac{1}{6} w x^3 - M_A \cdot x\right) + \cancel{C_1}$

Apply $y'(0) = 0 \Rightarrow C_1 = 0$

$y(x) = \frac{1}{EI}\left(\frac{1}{6} A_y x^3 - \frac{1}{24} w x^4 - \frac{1}{2} M_A \cdot x^2\right) + \cancel{C_2}$

Apply $y(0) = 0 \Rightarrow C_2 = 0$

Apply $y(L) = 0$ (due to roller at B)

$y(L) = 0 = \frac{1}{EI}\Big(\underbrace{\frac{A_y L^3}{6} - \frac{wL^4}{24} - \frac{M_A \cdot L^2}{2}}_{=0}\Big)$

Multiply by $\frac{24EI}{L^2}$ to get:

$4A_y L - wL^2 - 12M_A = 0$
eqn. (3)

$$\boxed{\begin{aligned} A_y + B_y &= w \cdot L \\ B_y L + M_A &= \frac{wL^2}{2} \\ 4A_y L - 12 M_A &= wL^2 \end{aligned}}$$

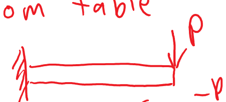
System of 3 eqns. in 3 unknowns

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & L & 1 \\ 4L & 0 & -12 \end{bmatrix} \begin{Bmatrix} A_y \\ B_y \\ M_A \end{Bmatrix} = \begin{Bmatrix} wL \\ wL^2/2 \\ wL^2 \end{Bmatrix}$$

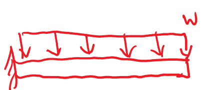
$$\begin{Bmatrix} A_y \\ B_y \\ M_A \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & L & 1 \\ 4L & 0 & -12 \end{bmatrix}^{-1} \begin{Bmatrix} wL \\ \frac{1}{2}wL^2 \\ wL^2 \end{Bmatrix} = \begin{Bmatrix} (5/8)wL \\ (3/8)wL \\ (1/8)\cdot wL^2 \end{Bmatrix}$$

Superposition Method

From table



$$y_{max} = \delta_B = \frac{-PL^3}{3EI}$$



$$y_{max} = \delta_B = \frac{-wL^4}{8EI}$$

Geometric Compatibility

$$y_B = 0$$

Set $P \to -B_y$

b.c. P acts down in the table, but B_y acts up in our F.B.D.

$$y_B = (\delta_B)_1 + (\delta_B)_2 = 0$$

$$\frac{24EI}{L^3}\left(\frac{-(-B_y)\cdot L^3}{3EI} + \frac{-wL^4}{8EI} \right) = 0$$

$$8B_y - 3wL = 0$$

$$\boxed{B_y = \frac{3wL}{8}}$$

From eqn (1):

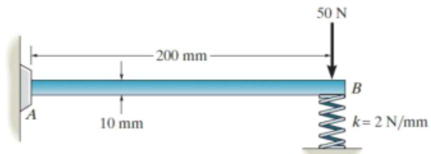
$$A_y + \left(\frac{3wL}{8}\right) - wL = 0$$

$$\boxed{A_y = \frac{5wL}{8}}$$

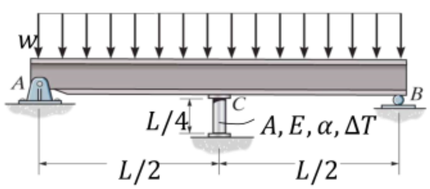
From eqn. (2)

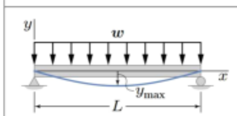
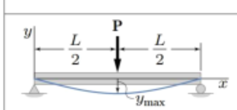
$$\left(\frac{3wL}{8}\right)L + M_A - \frac{wL^2}{2} = 0$$

$$M_A = \frac{4wL^2}{8} - \frac{3wL^2}{8} = \boxed{\frac{wL^2}{8}}$$



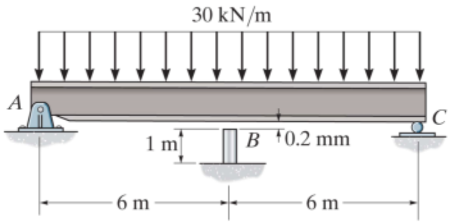
Determine the deflection at B



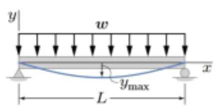
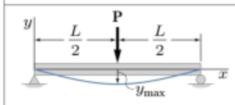
	y_{max} $-\frac{5wL^4}{384EI}$
	y_{max} $-\frac{PL^3}{48EI}$

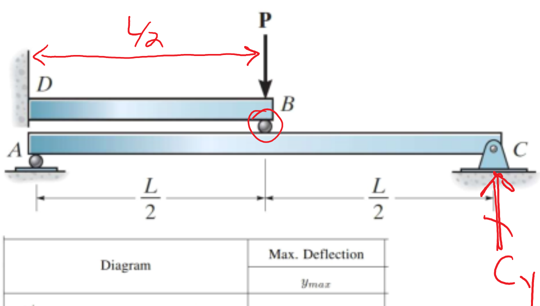
The beam is supported by a pin at A, a roller at B, and a deformable post at C. The post has length L/4, cross-sectional area A, modulus of elasticity E, and thermal expansion coefficient α. The beam has constant moment of inertia $I = A\,L^2$ and modulus of elasticity E (the same as the post).

(a) Determine the internal force in the post when the distributed load w is applied AND the post experiences an increase in temperature ΔT, where $\Delta T > 0$.



Before the uniform distributed load is applied on the beam, there is a small gap of 0.2 mm between the beam and the post at B. Determine the support reactions at A, B, and C. The post at B has a diameter of 40 mm, and the moment of inertia of the beam is $I = 875 \times 10^6\ \text{mm}^4$. The post and the beam are made of material having a modulus of elasticity of $E = 200$ GPa.

	y_{max} $-\frac{5wL^4}{384EI}$
	y_{max} $-\frac{PL^3}{48EI}$



Find the reaction force at point C using superposition methods

Geometric compatibility
Match deflections at B.

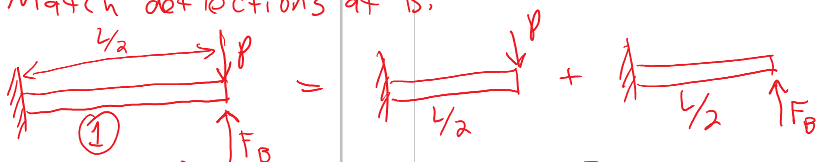
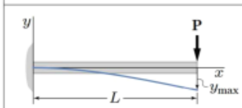
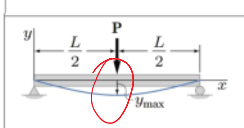
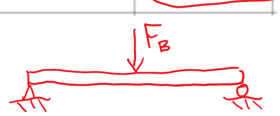


Diagram	Max. Deflection y_{max}
	$-\frac{PL^3}{3EI}$
	$-\frac{PL^3}{48EI}$

Find $(\delta_B)_1 = \frac{-P(L/2)^3}{3EI} + \frac{F_B \cdot (L/2)^3}{3EI} = \frac{(F_B - P)L^3}{24EI}$

Find $(\delta_B)_2$ (set $P \to F_B$)

$(\delta_B)_2 = \frac{-F_B L^3}{48EI}$



Geom. Compat.

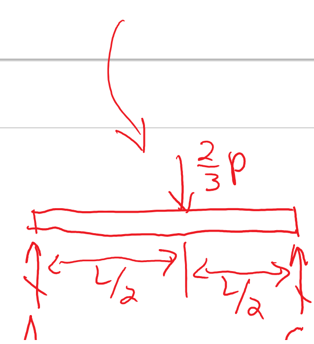
$(\delta_B)_1 = (\delta_B)_2$

$\left(\frac{(F_B - P)L^3}{24 \cdot EI} = \frac{-F_B L^3}{48EI}\right)\frac{48EI}{L^3}$

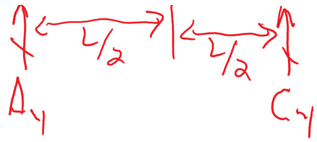
$2(F_B - P) = -F_B$

$2F_B - 2P = -F_B$

$F_B = \frac{2}{3}P$



$F_B = \frac{\;}{3} P$



$$(\Sigma M)_A = 0 \Rightarrow -\frac{\cancel{2}}{3} P \cdot \frac{\cancel{L}}{\cancel{2}} + C_y \cdot \cancel{L} = 0$$

$$-\frac{P}{3} + C_y = 0$$

$$\boxed{C_y = \frac{P}{3}}$$