Chapter 5: Torsion

Chapter Objectives

- ✓ Determine the shear stresses in a circular shaft due to torsion
- ✓ Determine the angle of twist
- ✓ Analyze statically indeterminate torque-loaded members also determinate
- ✓ Analyze stresses for inclined planes
- ✓ ~~Deal with thin-walled tubes~~

Torsion of shafts

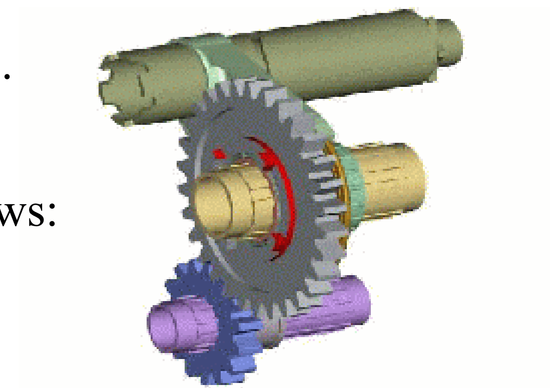
twisting moment

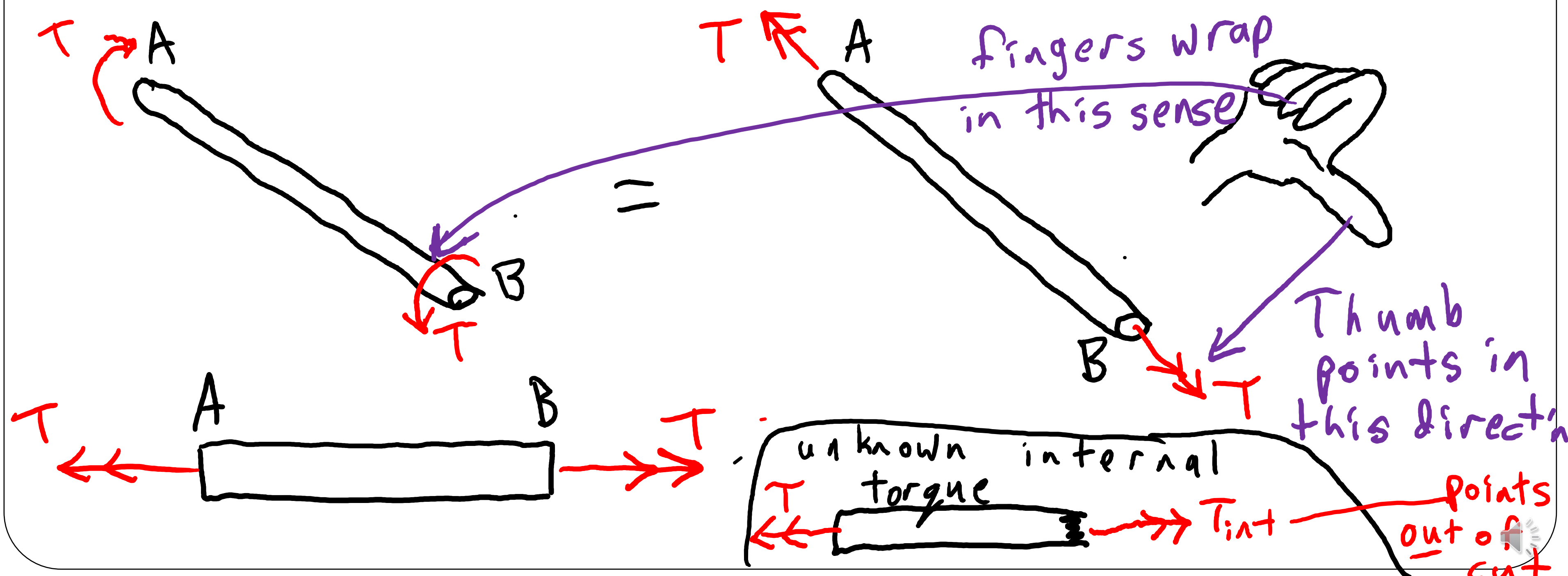
- Refers to the twisting of a specimen when it is loaded by couples (or moments) that produce rotation about the longitudinal axis.
- Applications: aircraft engines, car transmissions, bicycles, etc.
- Units: Force × distance [lb.in] or [N.m]
- Torques are vector quantities ~~and may~~ be represented as follows:

dimensions

units

N-mm
lb-ft

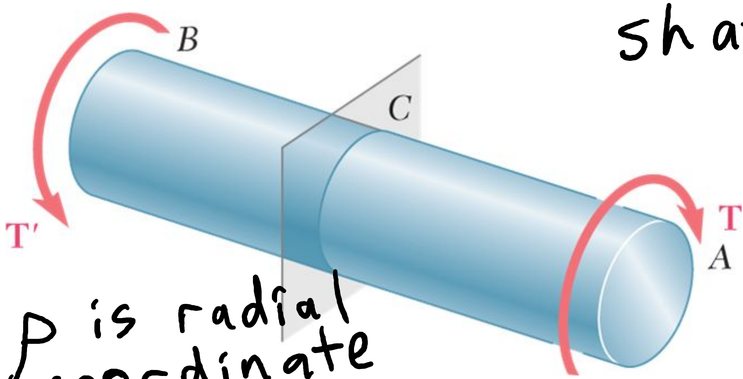




Equilibrium

(to find stress distribution inside shaft)

left of cut



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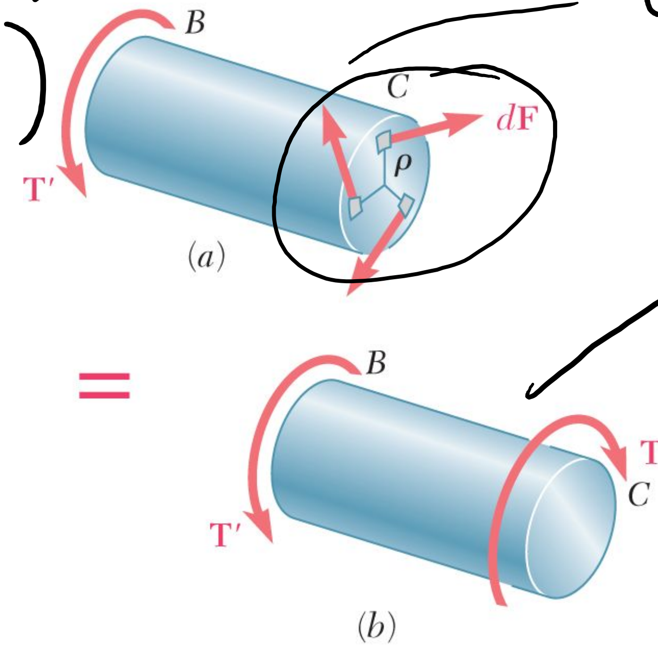
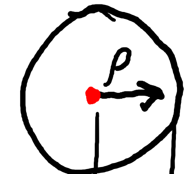


(a)

(b)

right of cut



ρ = 0 at center

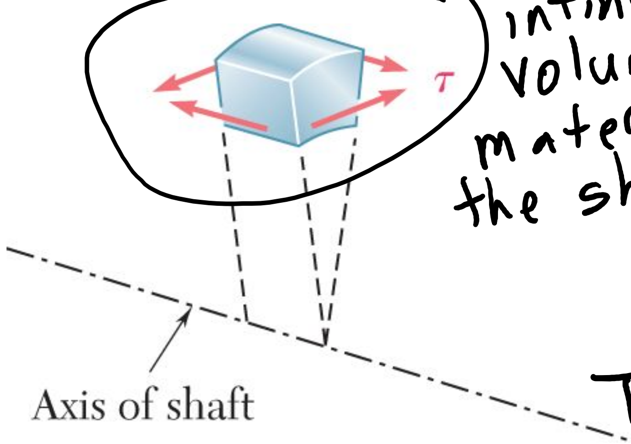
ρ = r

r is radius of the shaft

ρ is radial coordinate

$$\boxed{T = \int \rho \, dF} = \int \rho \underbrace{\tau \, dA}_{dF}$$

Although the net torque is known, the distribution of stresses is not.



infinitesimal volume of material in the shaft

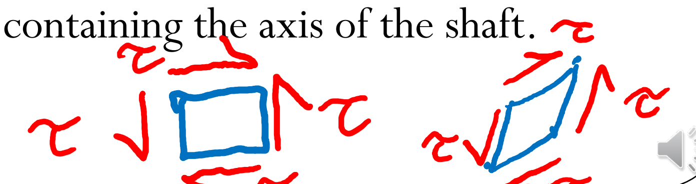
- Stress distribution is statically indeterminate—must consider shaft deformations

shear

we must use our intuition of the geometry of deformation

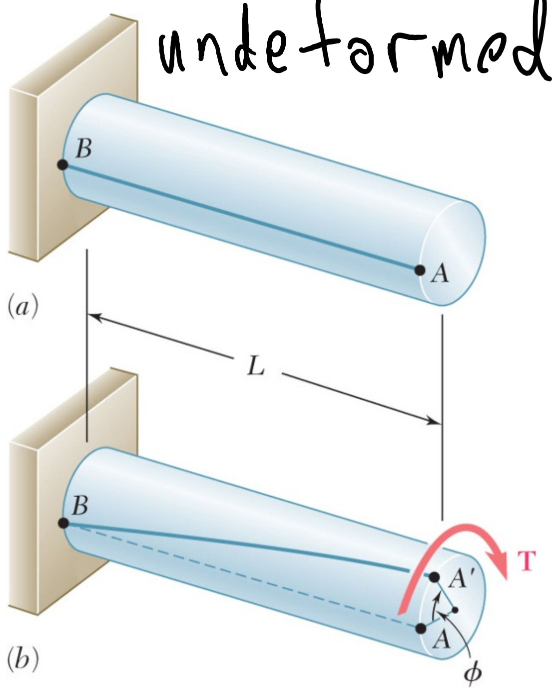
- Shear stress cannot exist in one plane only—equilibrium requires the existence of shear stresses on the faces formed by the two planes containing the axis of the shaft.

Top View of this element:



Shaft deformations

undeformed



Angle of twist: ϕ
Torque: T
Length: L
Diameter: d

From observation…

1) … the angle of twist of the shaft is:

A) proportional to the applied torque $\phi \propto T$

B) inversely proportional to the applied torque $\phi \propto \frac{1}{T}$

2) … the angle of twist of the shaft is:

A) proportional to the length $\phi \propto L$

B) inversely proportional to the length $\phi \propto \frac{1}{L}$

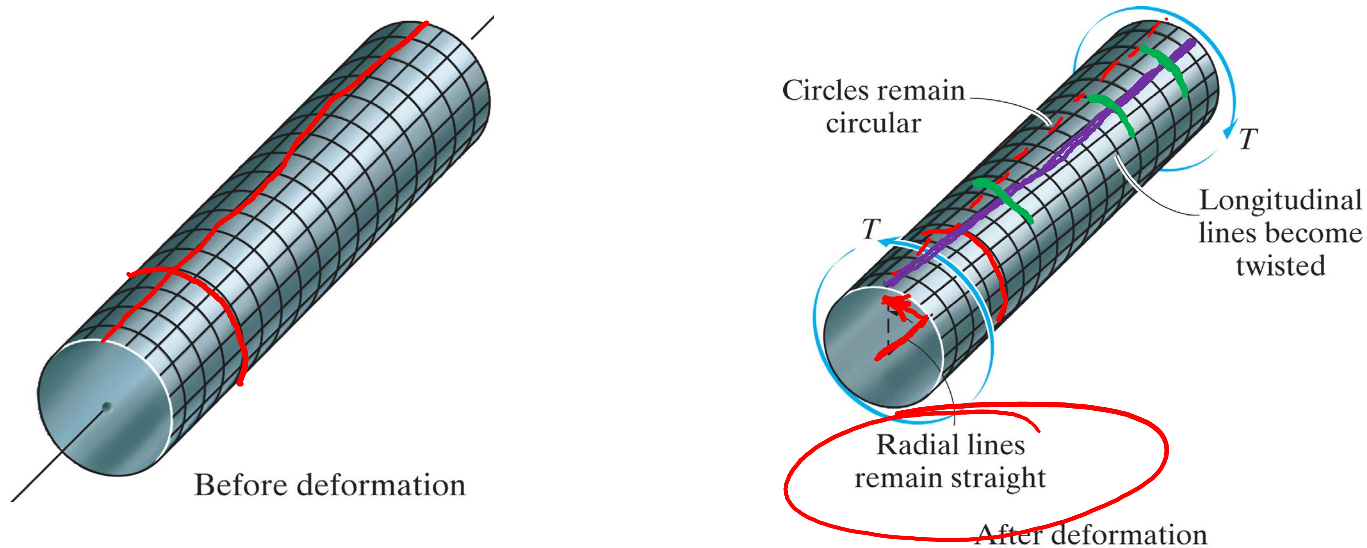
3) … the angle of twist of the shaft:

A) increases when the diameter of the shaft increases

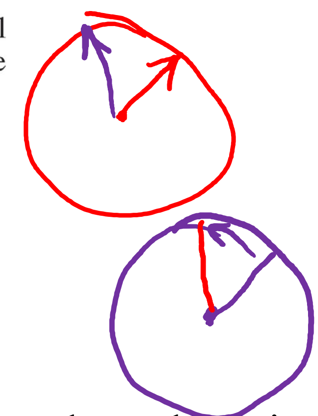
B) decreases when the diameter of the shaft increases

Assumptions made about torsion deformation:

- For circular shafts (hollow and solid): cross-sections remain plane and undistorted due to axisymmetric geometry
  - i.e. while different cross sections have distinct angles of twist, each one of them rotates as a solid rigid slab
  - Longitudinal lines twist into a helix that intersects the circular cross sections at equal angles

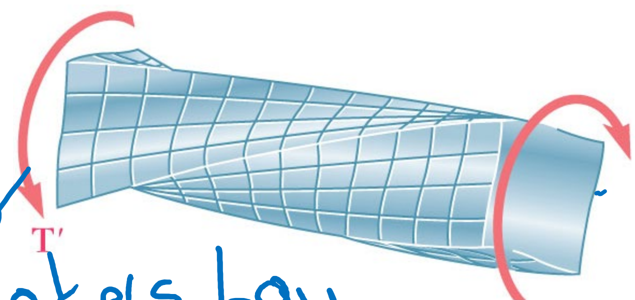


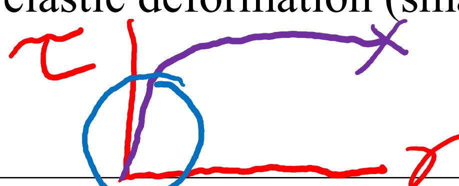
$$\frac{d\phi}{dx} = const.$$



- Non-circular cross-sections warp and do not remain plane – we do not analyze these in TAM 251

- Linear and elastic deformation (small strains)

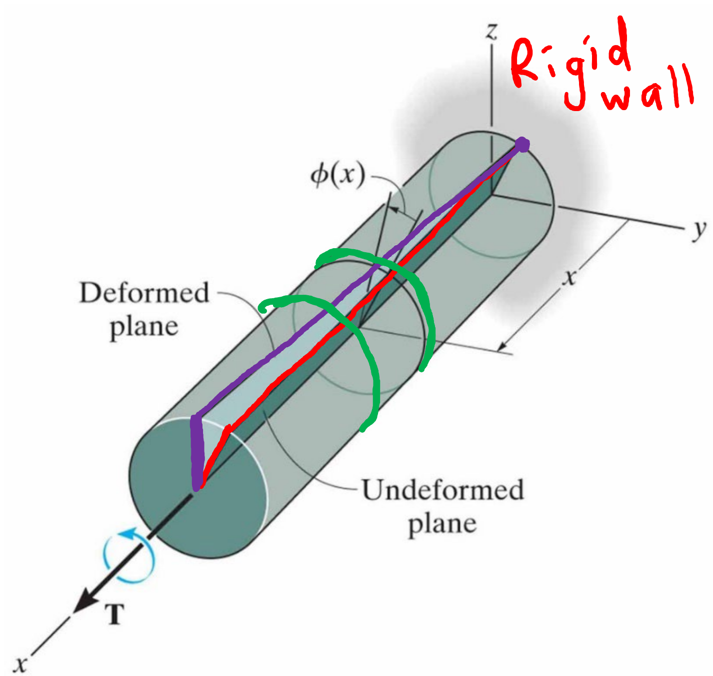


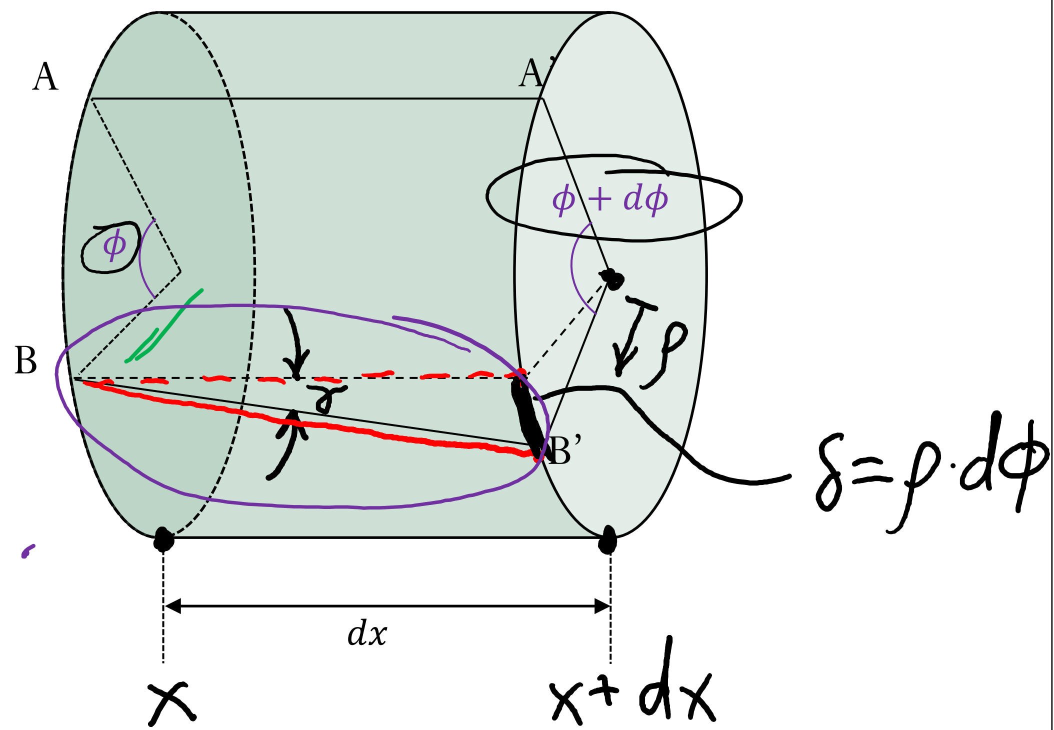


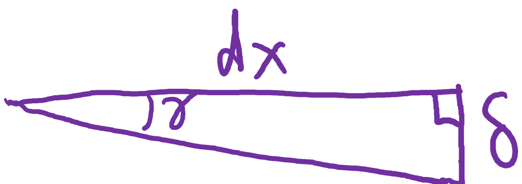
$\tau = G \cdot \gamma$

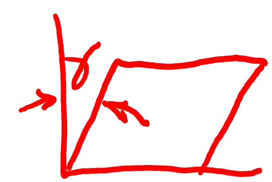
Hooke's Law

Shear strain – geometry of deformation









$$\tan\gamma = \frac{\delta}{dx} = \frac{\rho \cdot d\phi}{dx} = \rho \cdot \frac{d\phi}{dx}$$

if γ is small, then $\boxed{\gamma = \rho \cdot \frac{d\phi}{dx}}$

at $\rho = 0$
$\gamma = 0$

γ_{max} @ $\rho = R$

Shear stress distribution

1) Geometry: $\gamma = \rho \cdot \frac{d\phi}{dx}$

2) Constitutive: $\tau = G \cdot \gamma = G \cdot \rho \cdot \frac{d\phi}{dx}$

3) Equilibrium:

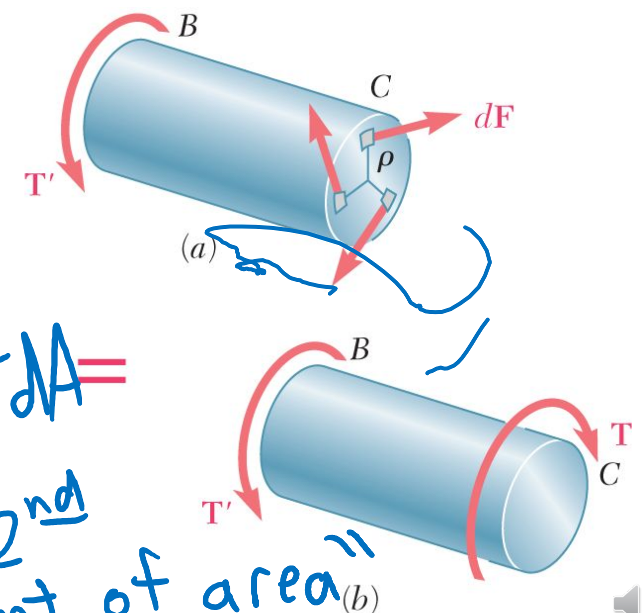
$$T = \int_A \rho \cdot dF = \int_A \rho \cdot \tau \cdot dA$$

$$= \int_A \rho \cdot \underbrace{\left(G \cdot \rho \cdot \frac{d\phi}{dx} \right)}_{\tau} dA$$

$$= G \cdot \frac{d\phi}{dx} \cdot \underbrace{\int_A \rho^2 \cdot dA}_{\text{purely geometric}}$$

$$\rightsquigarrow J = \int_A \rho^2 dA =$$

"polar $2^{\underline{nd}}$ moment of area"



Shear stress distribution

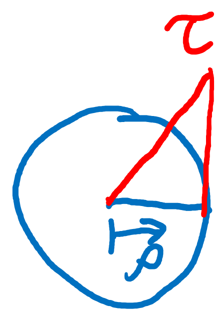
$$T = G \cdot J \cdot \frac{d\phi}{dx}$$

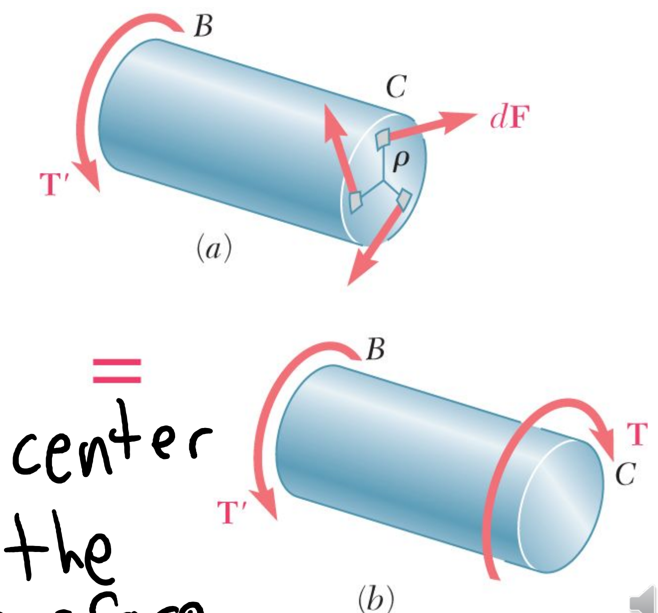
$$\text{-or-} \quad \frac{d\phi}{dx} = \frac{T}{G \cdot J}$$

$$\tau = G \cdot \rho \cdot \frac{d\phi}{dx}$$

$$\tau = \cancel{G} \cdot \rho \cdot \left(\frac{T}{\cancel{G} \cdot J}\right)$$

$$\boxed{\tau = \frac{T \cdot \rho}{J}}$$





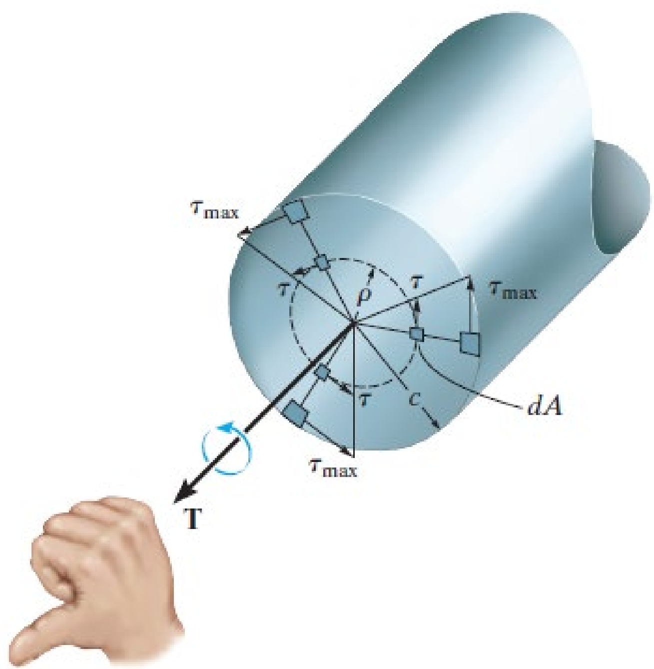
$\tau = 0$ at the center

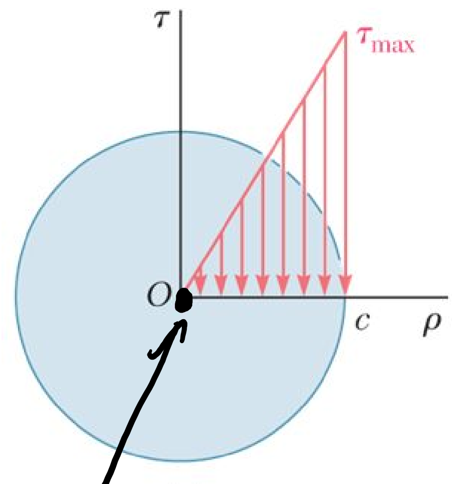
τ_{max} occurs on the outer surface

Shear stress in the elastic range

The shear stress varies linearly with the radial position in the section.

$$\boxed{\tau = \frac{T\,\rho}{J}}$$

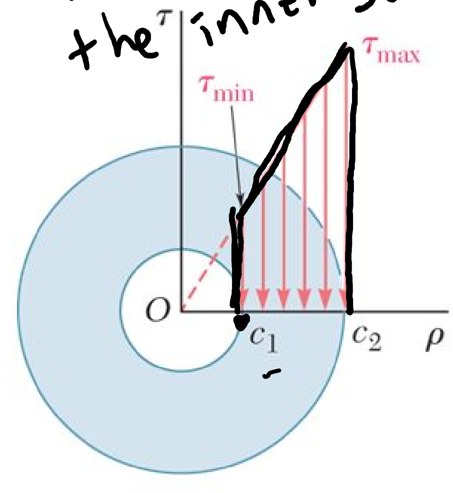




(a)

$\tau_{min} = 0$ at $\rho = 0$

$\tau_{min} \neq 0$, but is located at the inner surface



(b)

$\tau = 0$ @ $\rho = 0$ in fact it is undefined if there is no material at the center

Polar moment of ~~inertia~~ area

2nd

- Solid Shaft (radius R and diameter $D = 2R$):

$$\boxed{J = \frac{\pi}{2} R^4 = \frac{\pi}{32} D^4}$$

$$J = \int_A \rho^2 \cdot dA \, ; \quad dA = \rho \cdot d\rho \cdot d\theta$$

$$= \int_0^{2\pi} \int_{R_i}^{R_o} \rho^2 \cdot (\rho \cdot d\rho \cdot d\theta) = \int_0^{2\pi} \int_{R_i}^{R_o} \rho^3 \cdot d\rho \cdot d\theta = \frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

$$= 2\pi \int_{R_i}^{R_o} \rho^3 d\rho = \frac{2\pi}{4} \rho^4 \Big|_{R_i}^{R_o}$$

$$= \frac{\pi}{2}\left(R_o^4 - R_i^4\right)$$

- Hollow Shaft (inner radius R_i and outer radius R_o)

$$\boxed{J = \frac{\pi}{2}\left(R_o^4 - R_i^4\right) = \frac{\pi}{32}\left(D_o^4 - D_i^4\right)}$$

Angle of twist in the elastic range

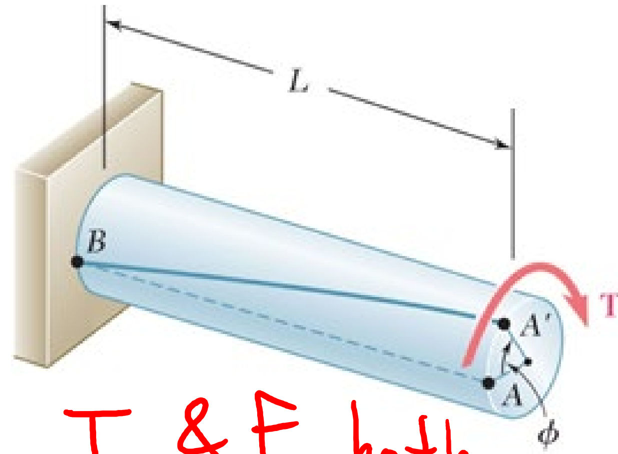
$$\frac{d\phi}{dx} = \frac{T}{G\,J}$$

For constant torque and cross-sectional area:

$$\int_0^L \frac{d\phi}{dx}\,dx = \int_0^L \underbrace{\frac{T}{G\,J}}_{\text{all const. in } x}\,dx$$

$$\phi \equiv \phi(L) - \phi(0) = \frac{T\,L}{G\,J} \qquad \delta = \frac{F\,L}{E\,A}$$

$$\boxed{\phi = \frac{T\,L}{G\,J}}$$



T & F both mech. loadings
G & E both material stiffnesses
J & A both geometric properties

Torsional stiffness: $k_T = \frac{G\,J}{L}$

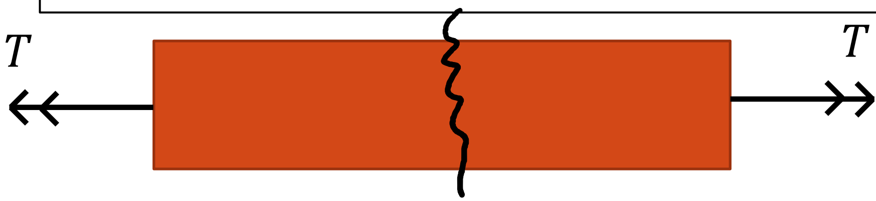
Torsional flexibility: $f_T = \frac{L}{G\,J}$

$T = k_T \cdot \phi \quad (F = k \cdot \delta)$

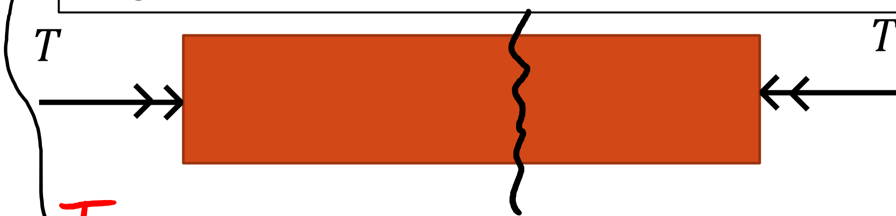
$\phi = f_T \cdot T$

Sign conventions

Positive torque (points outward from faces)



Negative torque (points inward towards faces)

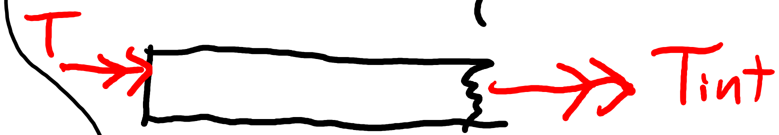


internal torque still drawn pointing <u>out</u> of the cut

$\sum T = 0$

$-T + T_{int} = 0$

$T_{int} = T$

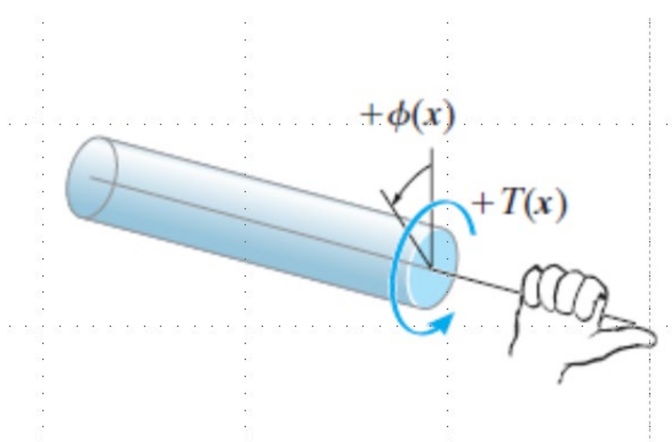


$\sum T = 0$

$T + T_{int} = 0$

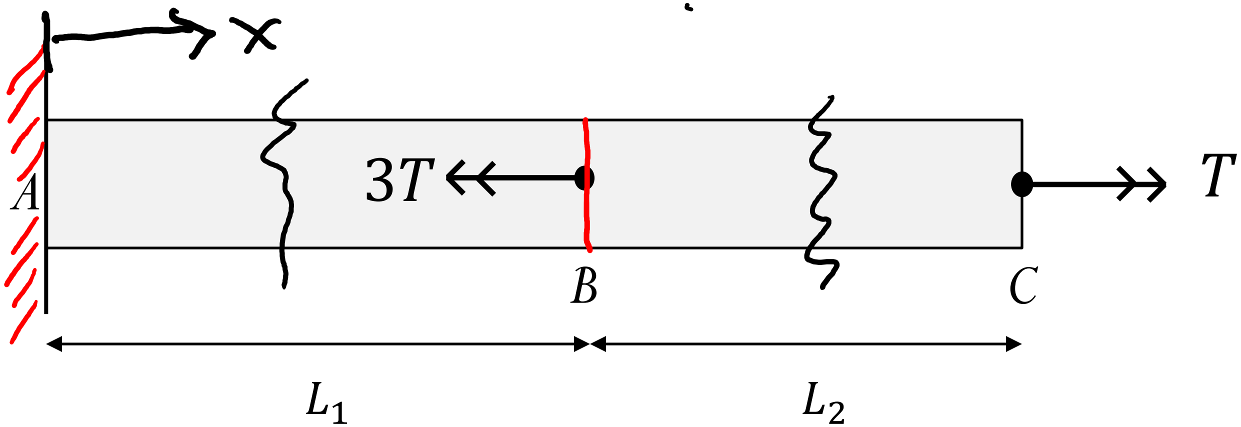
$T_{int} = -T$

Positive angle: CCW about torsion axis



Just as the torque follows the right-hand rule, so does the twist ϕ.

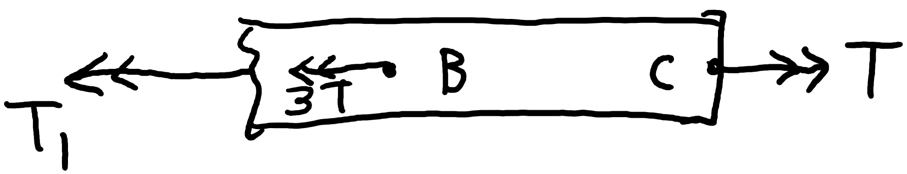
Example



Find: ϕ_B, ϕ_C

Given:

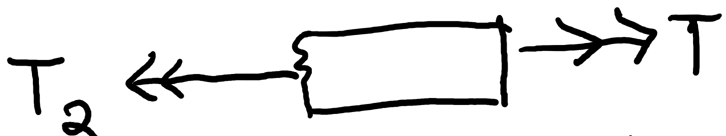
- shaft diameter D
- modulus G
- Lengths L_1 and L_2
- Applied torques at B and C



$\Sigma T = 0$

$\Rightarrow -T_1 - 3T + T = 0$

$T_1 = -2T$ } internal torque in $0 < x < L_1$



$\Sigma T = 0 \Rightarrow -T_2 + T = 0 \Rightarrow T_2 = T$

$\phi_{A/B} = \phi_B - \phi_A$

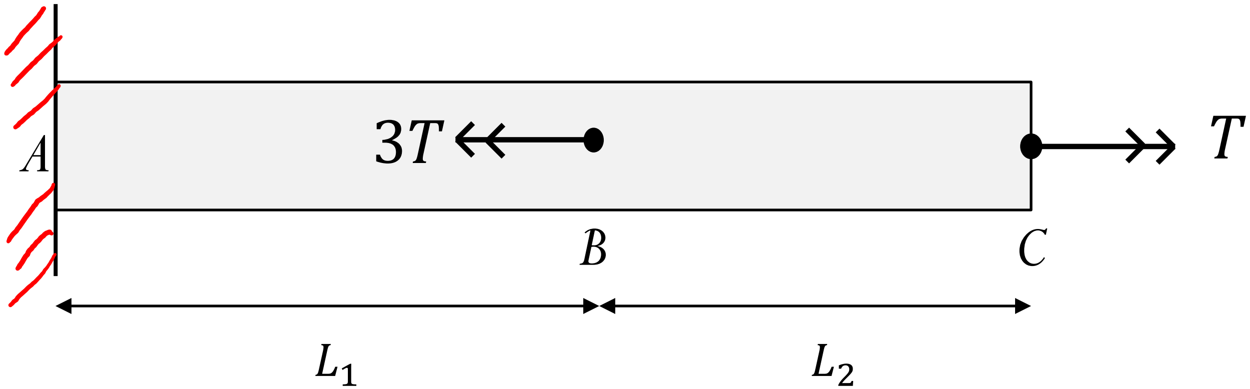
$\phi_A = 0$

$\phi_{A/B} = \phi_B$

$\phi_{A/B} = \phi_1 = \dfrac{T_1 L_1}{GJ}$

$\phi_B = \dfrac{-2T \cdot L_1}{G \cdot J}$

Example



<u>Find</u>: ϕ_B, ϕ_C

<u>Given</u>:

- shaft diameter D
- modulus G
- Lengths L_1 and L_2
- Applied torques at B and C

$$\phi_B = \frac{-2TL_1}{G\cdot J}$$

$$\phi_{C/B} = \phi_C - \phi_B$$

$$\phi_{C/B} = \phi_2 = \frac{T_2 \cdot L_2}{G\cdot J} = \frac{T\cdot L_2}{G\cdot J}$$

$$\phi_C = \phi_{C/B} + \phi_B = \frac{T\cdot L_2}{GJ} + \left(\frac{-2TL_1}{G\cdot J}\right) = \boxed{\frac{T}{GJ}\left(L_2 - 2L_1\right)}$$

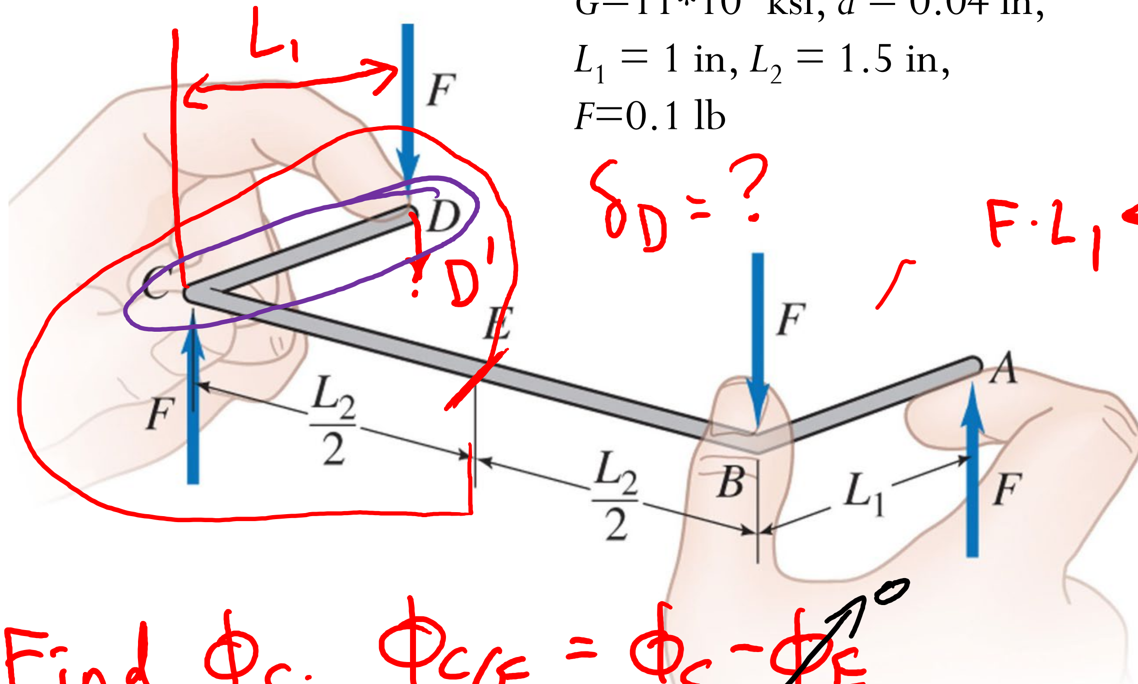
$$|\tau_{max}| = \frac{|T_{max}|\cdot D/2}{J} = \frac{|-2T|\cdot D}{2\cdot J} = \frac{TD}{J}$$

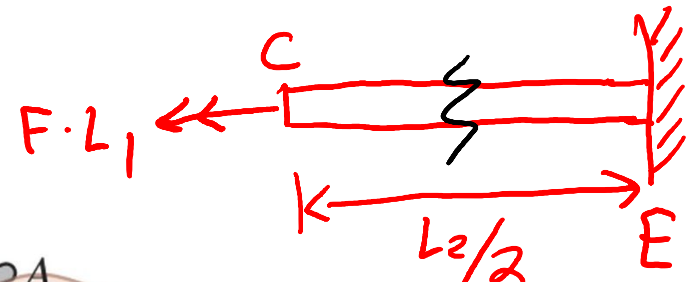
Example 1:

The bent steel wire is twisted by the four forces. Neglect bending of AB and CD due to the force F, and take B and C to have zero displacement. Assume the center plane E does not rotate, so A and D displace by equal amounts in the opposite directions. The wire has shear modulus G and diameter d. Determine the displacement of point D.

$G = 11*10^3$ ksi, $d = 0.04$ in,
$L_1 = 1$ in, $L_2 = 1.5$ in,
$F = 0.1$ lb

$\delta_D = ?$



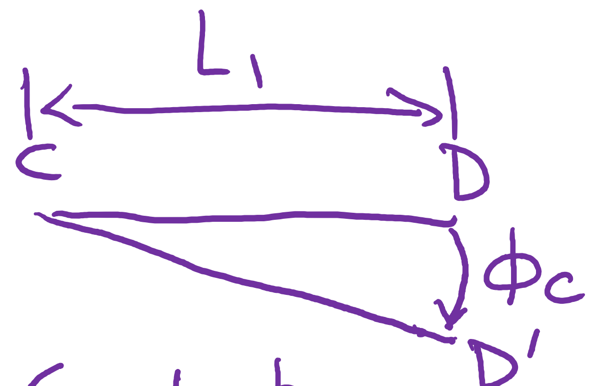


Find ϕ_C. $\phi_{C/E} = \phi_C - \cancel{\phi_E}^{\,0}$

$\Sigma T = 0$

$T = F \cdot L_1$

$$\phi_C = \frac{T \cdot L_2}{G J} = \frac{F \cdot L_1 \cdot L_2}{G \cdot \frac{\pi}{32} d^4}$$



$\delta_D = L_1 \cdot \phi_C$ assuming small ϕ_C

$$\delta_D = \frac{F \cdot L_1^2 \cdot L_2}{G \, \pi/32 \, d^4}$$

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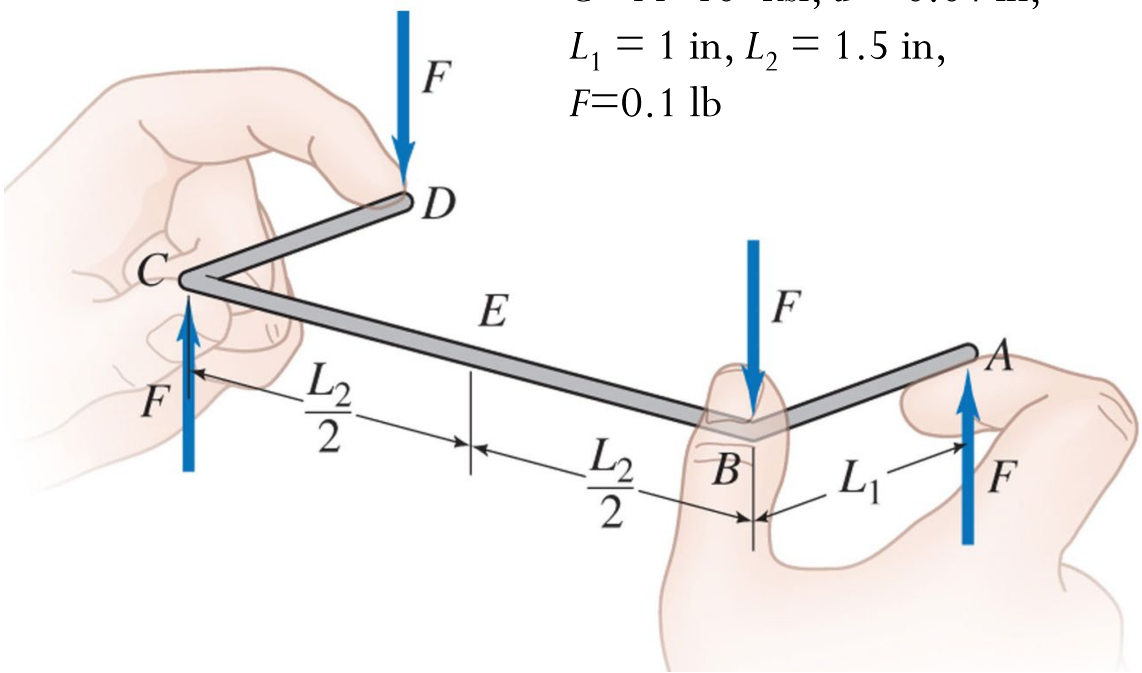
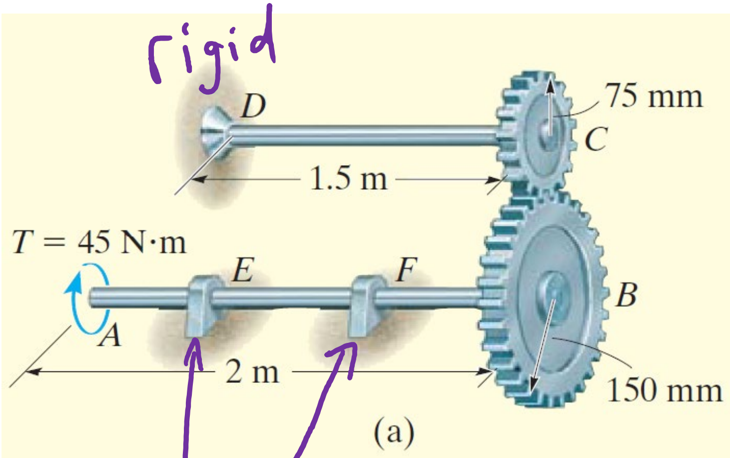


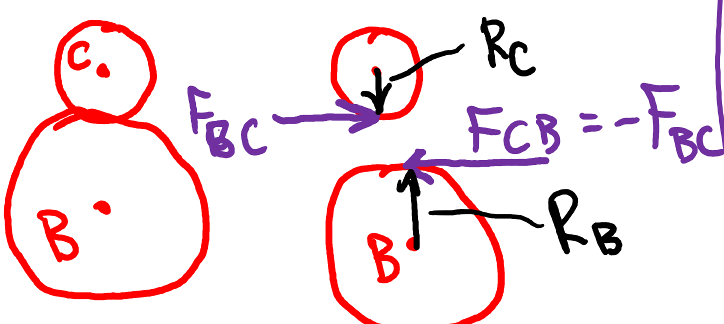
Gear systems with applied torque



Journal bearings

Two solid steel shafts are connected through gears at points B and C. The top shaft is mounted to a fixed wall at point D. Determine the angle of twist at point A assuming a torque of $T = 45$ N·m is applied at A. Each shaft has a diameter of $d = 20$ mm and a shear modulus of $G = 80$ GPa

How does torque transfer through the gears? N3L



How do the gears rotate?

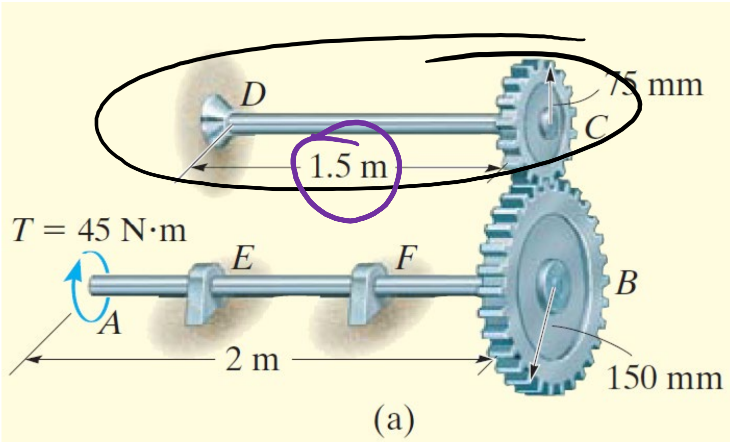
ϕ_B will have opposite sign as ϕ_C

Arclength s is same for both

$$|s| = |\phi_C \cdot r_C| = |\phi_B \cdot r_B|$$

$$\phi_C \cdot r_C = -\phi_B \cdot r_B$$

Gear systems with applied torque



(a)

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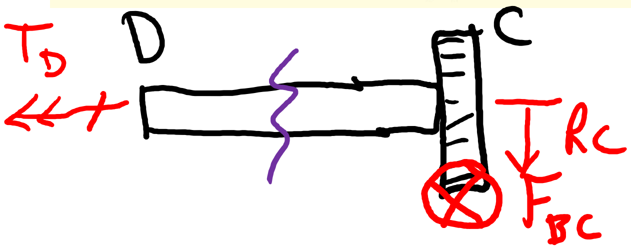


$\Sigma T = 0$

$-T_D + F_{BC} \cdot R_C = 0$

$F_{BC} = \frac{T_D}{R_C} \Rightarrow -T + \left(\frac{T_D}{R_C}\right) R_B = 0$

$\Rightarrow T = T_D \cdot \frac{R_B}{R_C}$

$T_D = T \cdot \frac{R_C}{R_B}$

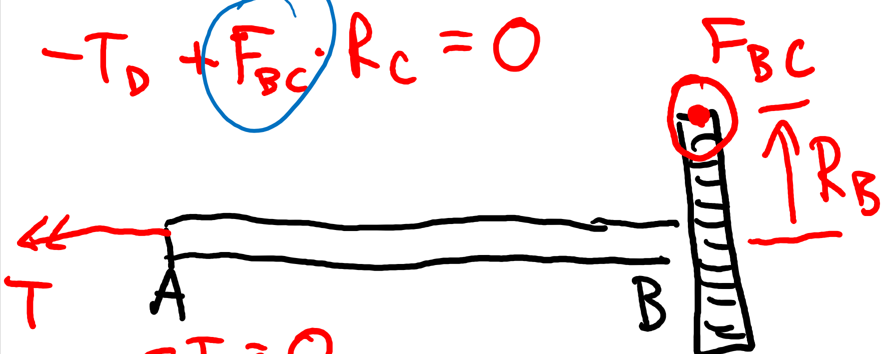
At D: $\phi_D = 0$

At C: $\phi_{C/D} = \phi_C - \phi_D = \phi_C$

$\phi_C = \frac{(T \cdot R_C/R_B)\, L_1}{G \cdot J}$

$\phi_B R_B = -\phi_C R_C$

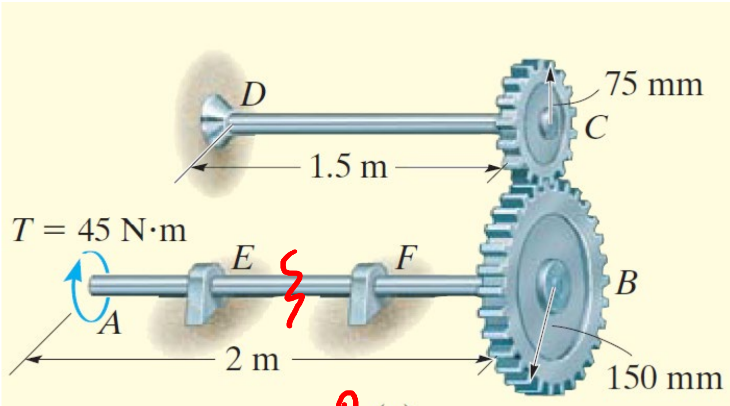
$\phi_B = -\phi_C \cdot \frac{R_C}{R_B}$



$\Sigma T = 0$

$\Rightarrow -T + F_{BC} \cdot R_B = 0$

Gear systems with applied torque



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$$\phi_C = \frac{T \cdot \frac{R_C}{R_B} \cdot L_1}{G \cdot J}$$

$$T_2 = T$$

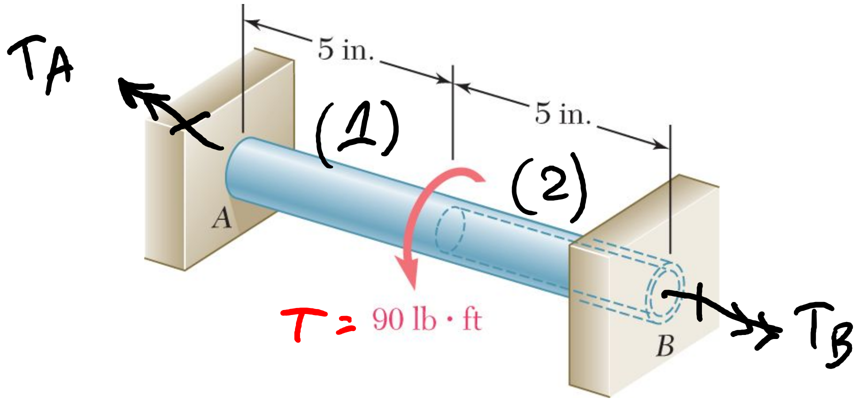
$$\phi_B = -\phi_C \cdot \frac{R_C}{R_B} = -\frac{T \cdot \left(\frac{R_C}{R_B}\right)^2 \cdot L_1}{G \cdot J}$$

$$\phi_{A/B} = \phi_A - \phi_B$$

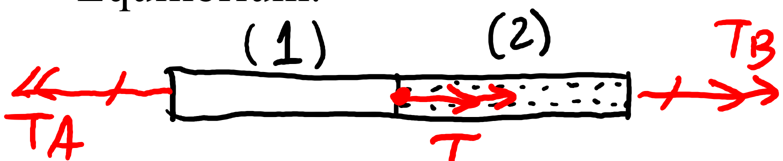
$$\phi_A = \phi_{A/B} + \phi_B = \frac{\overset{T}{\cancel{T_2}} \cdot L_2}{G \cdot J} + \left(\frac{-T\left(\frac{R_C}{R_B}\right)^2 \cdot L_1}{G \cdot J}\right)$$

$$\phi_A = \frac{T \cdot L_2}{G J} - \frac{T\left(R_C/R_B\right)^2 L_1}{G \cdot J} = \frac{T}{GJ}\left[L_2 - L_1\left(\frac{R_C}{R_B}\right)^2\right]$$

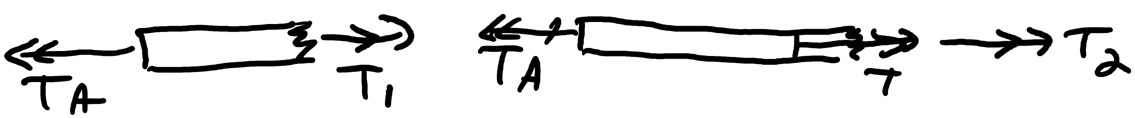
Statically Indeterminate Shafts



- Equilibrium:



$$\Sigma T = 0 \Rightarrow -T_A + T + T_B = 0$$



<u>Torque - Twist</u>

$$\phi = \frac{T \cdot L}{GJ}$$

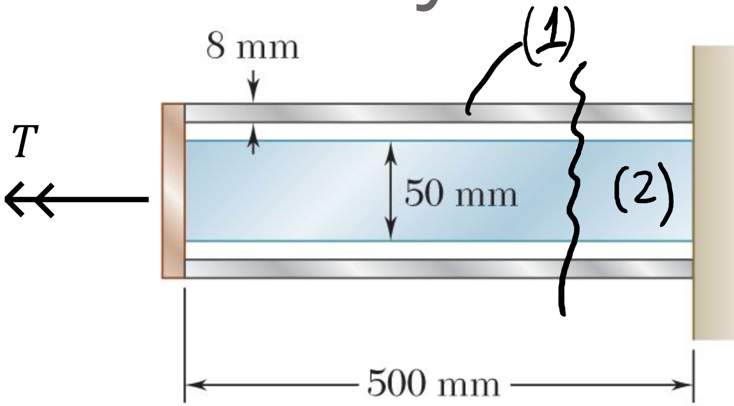
$$\phi_1 = \frac{T_1 \cdot L}{G J_1}$$

$$\phi_2 = \frac{T_2 \cdot L}{G \cdot J_2}$$

- Geometry of deformation:

$$\phi_1 + \phi_2 = 0 \Rightarrow \frac{T_1 \cdot L}{G \cdot J_1} + \frac{T_2 \cdot L}{G J_2} = 0$$

Statically Indeterminate Shafts



Torque – Twist

$$\phi_1 = \frac{T_1 \cdot L}{G_1 \cdot J_1}$$

$$\phi_2 = \frac{T_2 \cdot L}{G_2 J_2}$$

- Equilibrium:

$\Sigma T = 0$

$-T + T_1 + T_2 = 0 \Rightarrow T_1 + T_2 = T$

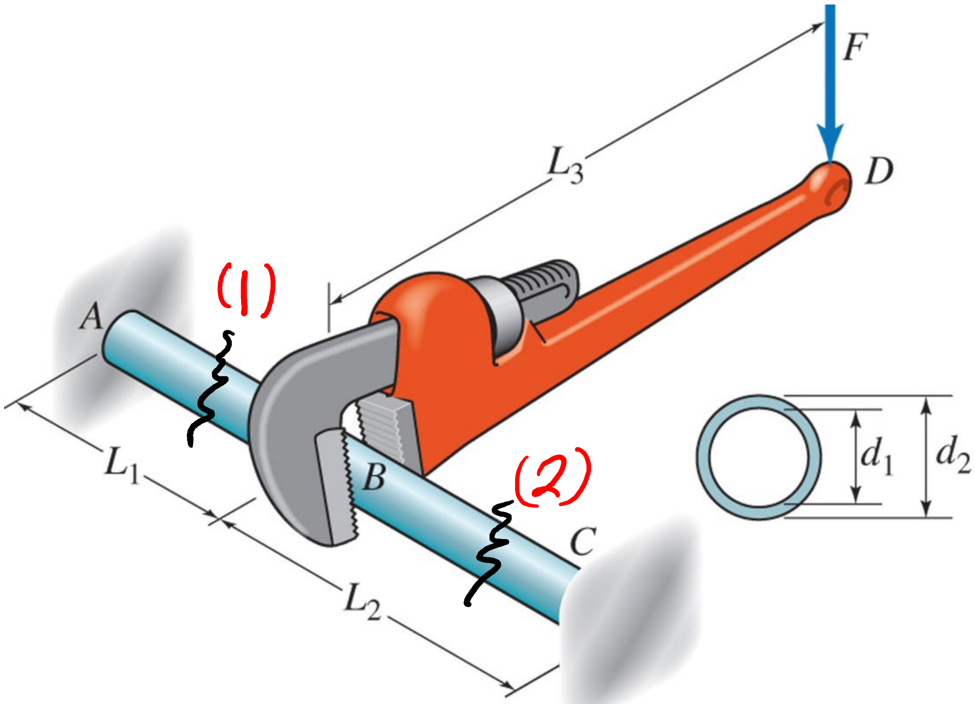
- Geometry of deformation:

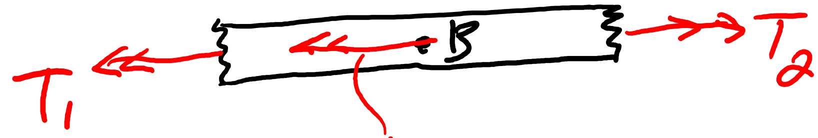
Rigid end cap $\Rightarrow \phi_1 = \phi_2 \Rightarrow \frac{T_1 \cdot \cancel{L}}{G_1 J_1} = \frac{T_2 \cdot \cancel{L}}{G_2 \cdot J_2}$

$T_2 = T - T_1 \Rightarrow \frac{T_1}{G_1 J_1} = \frac{(T - T_1)}{G_2 \cdot J_2}$ ... solve for T_1 then for T_2

Example 3:

The plastic tube ($G = 1$ GPa) is being twisted with the wrench. Say the ends A and C are fixed, and the tube is supported against bending. A force $F = 40$ N is applied perpendicularly to the length of the wrench. Determine (a) the displacement of the end D of the wrench and (b) the maximum shear stress in the tube. Treat the wrench as rigid. ($L_1 = 100$ mm, $L_2 = 150$ mm, $L_3 = 250$ mm, $d_1 = 24$ mm, and $d_2 = 30$ mm.)





$\sum T = 0 \Rightarrow -T_1 - FL_3 + T_2 = 0$

$\Rightarrow T_2 = T_1 + FL_3$

$\phi_{B/A} = \phi_B - \cancel{\phi_A}^{\,0} = \phi_B$

$\phi_{B/A} = \phi_1$

$\phi_{C/B} = \cancel{\phi_C}^{\,0} - \phi_B = -\phi_B$

$\phi_{C/B} = \phi_2$

$$\phi_1 = \frac{T_1 L_1}{GJ}$$

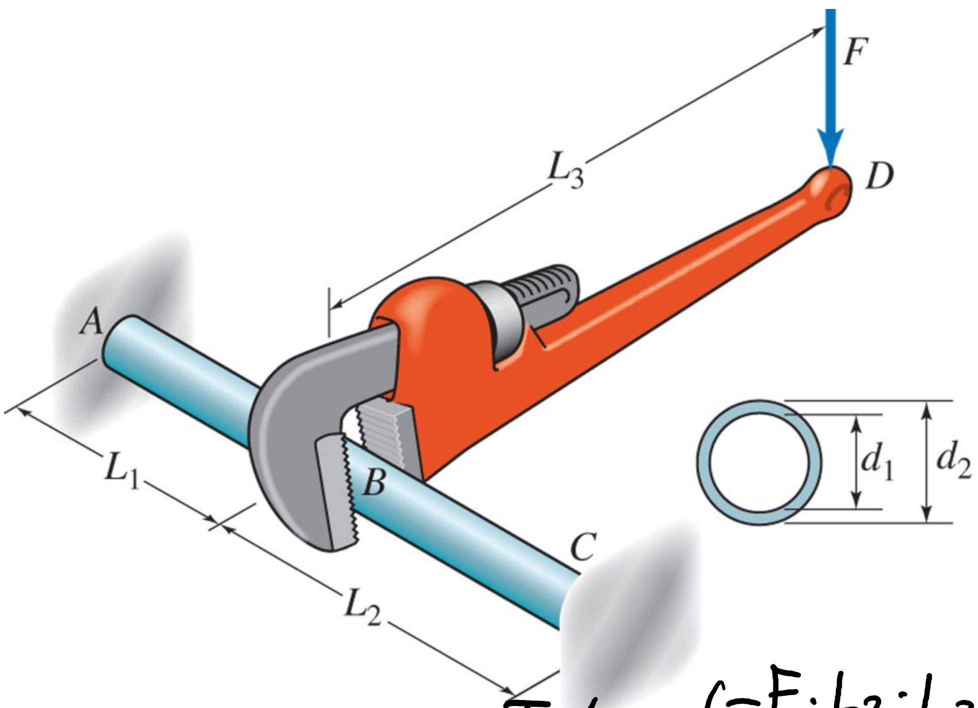
$$\phi_2 = \frac{T_2 L_2}{GJ}$$

$\Rightarrow \phi_1 + \phi_2 = 0 \Rightarrow \phi_1 = -\phi_2$

$\Rightarrow \frac{T_1 L_1}{\cancel{GJ}} = \frac{-T_2 L_2}{\cancel{GJ}}$

Example 3:

The plastic tube ($G = 1$ GPa) is being twisted with the wrench. Say the ends A and C are fixed, and the tube is supported against bending. A force $F = 40$ N is applied perpendicularly to the length of the wrench. Determine (a) the displacement of the end D of the wrench and (b) the maximum shear stress in the tube. Treat the wrench as rigid. ($L_1 = 100$ mm, $L_2 = 150$ mm, $L_3 = 250$ mm, $d_1 = 24$ mm, and $d_2 = 30$ mm.)



$$T_2 = T_1 + F L_3$$

$$T_1 L_1 = -T_2 L_2$$

$$T_1 L_1 = -(T_1 + F L_3) \cdot L_2$$

$$T_1 L_1 + T_1 L_2 = -F \cdot L_3 \cdot L_2$$

$$T_1 = \frac{-F \cdot L_2 \cdot L_3}{L_1 + L_2}$$

$$\phi_B = \phi_1 = \frac{T_1 L_1}{GJ} = \frac{\left(\frac{-F \cdot L_2 \cdot L_3}{L_1 + L_2}\right) \cdot L_1}{G \cdot J} = \frac{-F \cdot L_1 L_2 L_3}{GJ(L_1 + L_2)}$$

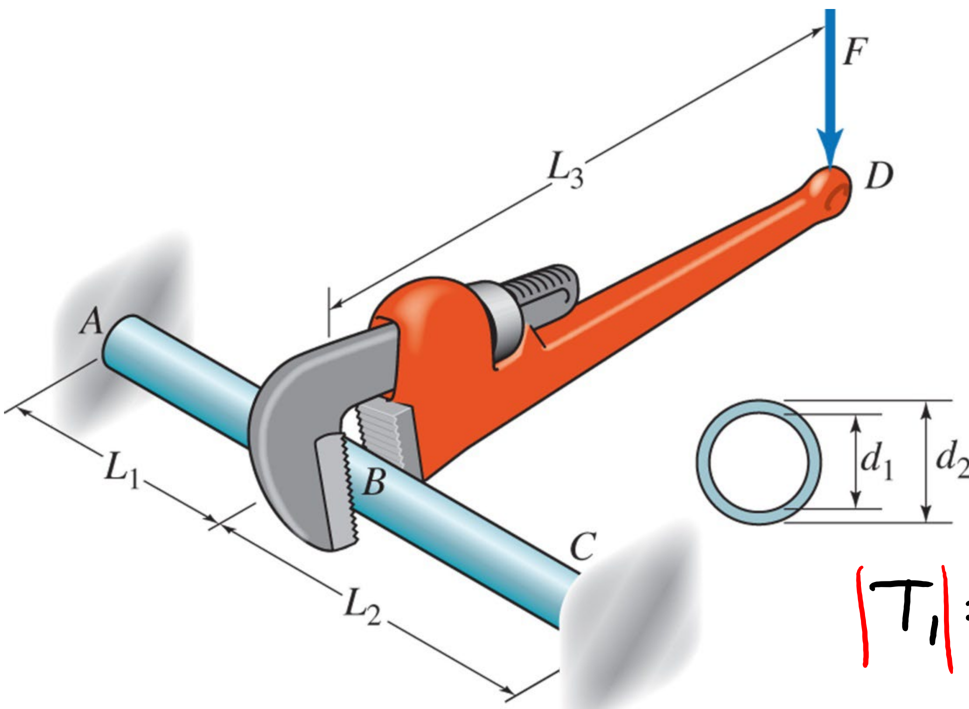
$$\delta_D = \phi_B \cdot L_3 = \frac{-F \cdot L_1 L_2 L_3^2}{GJ(L_1 + L_2)}$$

$$\tau_{max} = \frac{T_1 (d_2/2)}{J}$$

-or- $\tau_{max} \; \frac{T_2 (d_2/2)}{J}$

Example 3:

The plastic tube ($G = 1$ GPa) is being twisted with the wrench. Say the ends A and C are fixed, and the tube is supported against bending. A force $F = 40$ N is applied perpendicularly to the length of the wrench. Determine (a) the displacement of the end D of the wrench and (b) the maximum shear stress in the tube. Treat the wrench as rigid. ($L_1 = 100$ mm, $L_2 = 150$ mm, $L_3 = 250$ mm, $d_1 = 24$ mm, and $d_2 = 30$ mm.)



τ_{max} in shaft (1) or (2)?

$$\tau_{max} = \frac{T \cdot d_2/2}{J}$$

Which is greater?

$|T_1|$ or $|T_2|$?

$$|T_1| = \left|\frac{-F \cdot L_2 L_3}{L_1 + L_2}\right|$$

$$T_2 = T_1 + F \cdot L_3 = \frac{-F L_2 L_3}{L_1 + L_2} + F \cdot L_3 = F \cdot L_3 \cdot \left(1 - \frac{L_2}{L_1 + L_2}\right)$$

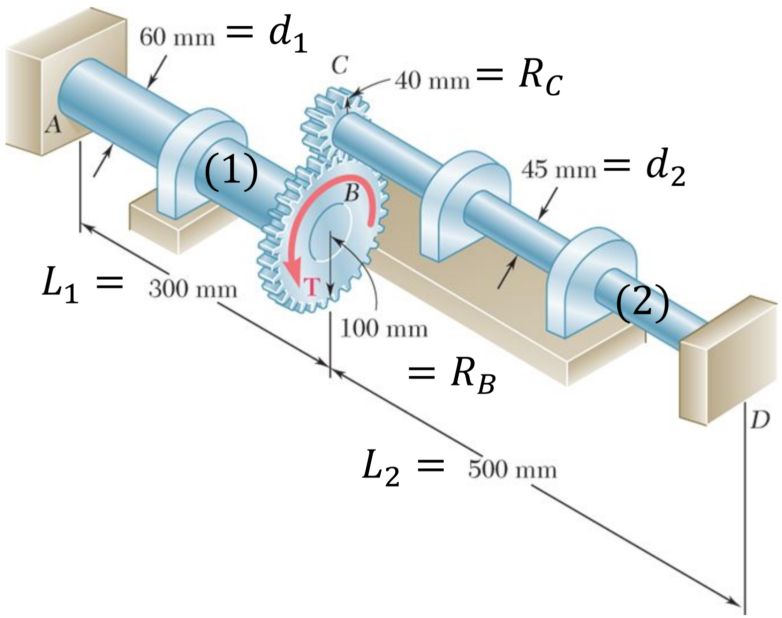
$$|T_2| = \left|F \cdot L_3 \cdot \left(\frac{L_1 + \cancel{L_2} - \cancel{L_2}}{L_1 + L_2}\right)\right| = \left|\frac{F \cdot L_3 L_1}{L_1 + L_2}\right|$$

$L_2 > L_1$

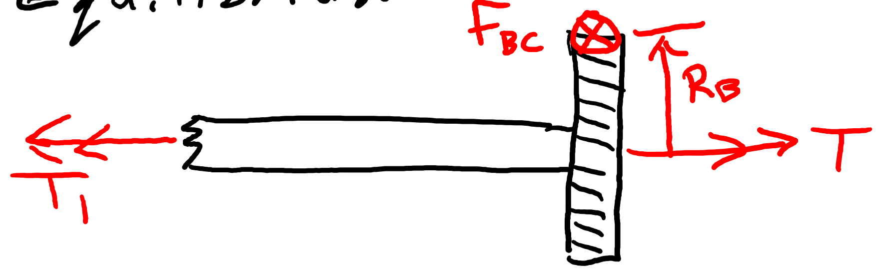
$\Rightarrow T_2 > T_1$

$$\tau_{max} = \frac{T_2 \cdot d_2}{2J}$$

Example 4: Given $T = 4\ kN \cdot m$, determine the maximum shear stress in shaft AB.



Equilibrium



$$\Sigma T = 0 \Rightarrow -T_1 + T - F_{BC} \cdot R_B = 0$$

$$F_{BC} = \frac{T - T_1}{R_B}$$

$$\Sigma T = 0$$

$$-F_{BC} \cdot R_c + T_2 = 0$$

$$\Rightarrow -\left(\frac{T - T_1}{R_B}\right) R_c + T_2 = 0$$

Torque - Twist

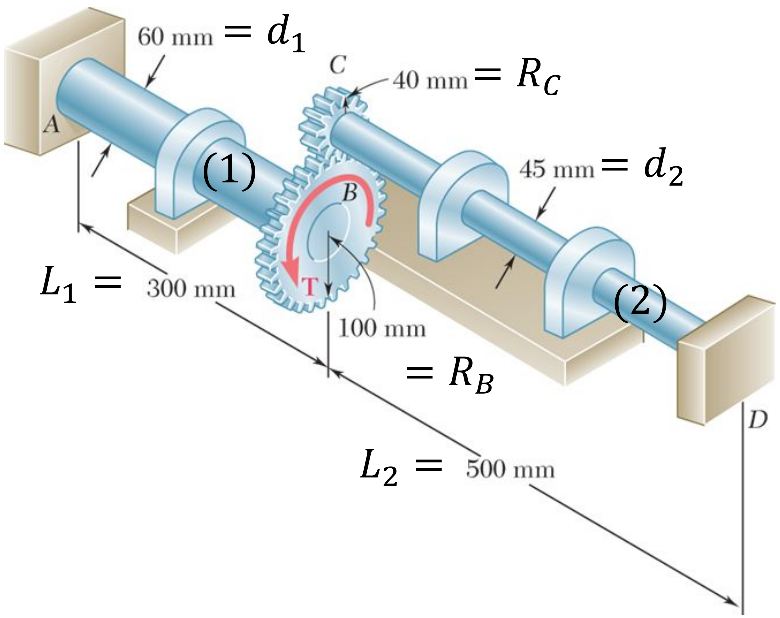
$$\phi_1 = \phi_{B/A} = \phi_B - \cancel{\phi_A}^{\,0}$$

$$\phi_B = \frac{T_1 L_1}{G J_1}$$

$$\phi_2 = \phi_{D/C} = \cancel{\phi_D}^{\,0} - \phi_C$$

$$\phi_C = -\phi_2 = -\frac{T_2 \cdot L_2}{G \cdot J_2}$$

Example 4: Given $T = 4\ kN \cdot m$, determine the maximum shear stress in shaft AB.



$$-\left(\frac{T - T_1}{R_B}\right) R_C + T_2 = 0 \quad ★$$

$$\phi_B = \frac{T_1 L_1}{G J_1} \qquad \phi_C = \frac{-T_2 L_2}{G J_2}$$

Geometric compatibility

$$\phi_B R_B = -\phi_C R_C$$

$$\frac{T_1 L_1 R_B}{\cancel{G} J_1} = \cancel{-}\left(\frac{\cancel{-}T_2 L_2}{\cancel{G} J_2}\right) \cdot R_C$$

Solve for T_2 & plug it into ★:

$$T_2 = \frac{T_1 L_1 R_B}{J_1} \cdot \frac{J_2}{L_2 R_C}$$

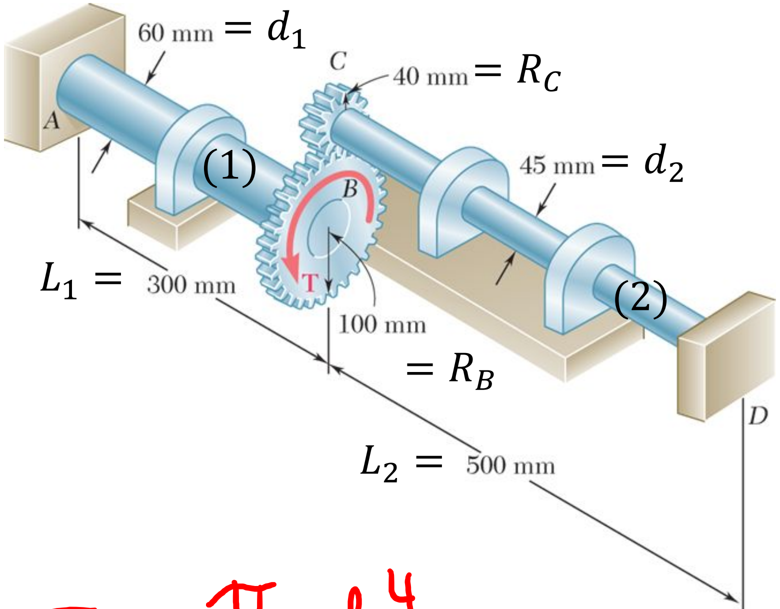
$$T_2 = T_1 \left(\frac{L_1}{L_2}\right)\left(\frac{R_B}{R_C}\right)\left(\frac{J_2}{J_1}\right)$$

$$(T_1 - T)\frac{R_C}{R_B} + T_1\left(\frac{L_1}{L_2}\right)\left(\frac{R_B}{R_C}\right)\left(\frac{J_2}{J_1}\right) = 0$$

$$T_1\left[\frac{R_C}{R_B} + \frac{L_1}{L_2}\frac{R_B}{R_C}\frac{J_2}{J_1}\right] = T \cdot \frac{R_C}{R_B}$$

$$T_1\left(\frac{L_2 R_C^2 J_1 + L_1 R_B^2 J_2}{L_2 \cancel{R_B} R_C J_1}\right) = T \cdot \frac{R_C}{\cancel{R_B}}$$

Example 4: Given $T = 4\ kN \cdot m$, determine the maximum shear stress in shaft AB.



$$T_1 \cdot \left(\frac{L_2 R_C^2 J_1 + L_1 R_B^2 J_2}{L_2 R_C J_1} \right) = T \cdot R_C$$

$$T_1 = \frac{T \cdot L_2 \cdot R_C^2 \cdot J_1}{L_2 R_C^2 J_1 + L_1 R_B^2 J_2}$$

$$(\tau_{max})_{AB} = \frac{T_1 \cdot (d_1/2)}{J_1}$$

$$J_1 = \frac{\pi}{32} d_1^4$$

$$= \frac{T_1 \cdot d_1}{2(\frac{\pi}{32}) d_1^4} = \frac{16 \cdot T_1}{\pi \cdot d_1^3}$$

$$J_2 = \frac{\pi}{32} d_2^4$$

$$(\tau_{max})_{AB} = \frac{16}{\pi d_1^3} \left[\frac{T \cdot L_2 \cdot R_C^2 \cdot J_1}{L_2 R_C^2 J_1 + L_1 R_B^2 J_2} \right]$$

$$= \frac{T_1 \cdot d_1}{2 \cdot J_1} = \frac{d_1}{2} \left[\frac{T \cdot L_2 \cdot R_C^2}{L_2 R_C^2 J_1 + L_1 R_B^2 J_2} \right]$$