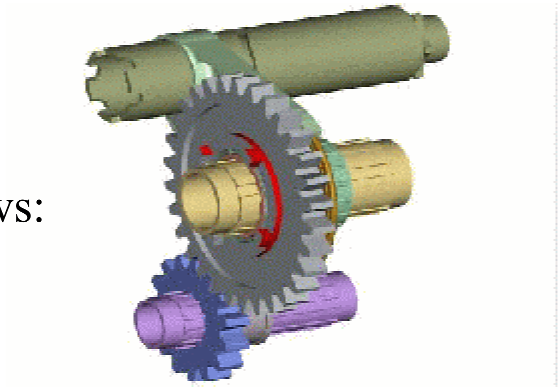
Chapter 5: Torsion

Chapter Objectives

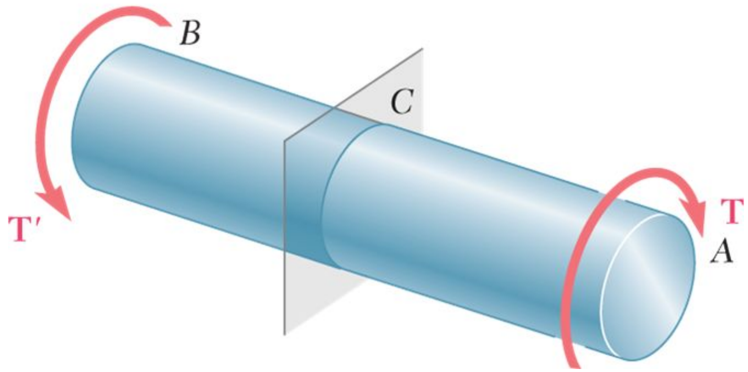
- ✓ Determine the shear stresses in a circular shaft due to torsion
- ✓ Determine the angle of twist
- ✓ Analyze statically indeterminate torque-loaded members
- ✓ Analyze stresses for inclined planes
- ✓ Deal with thin-walled tubes

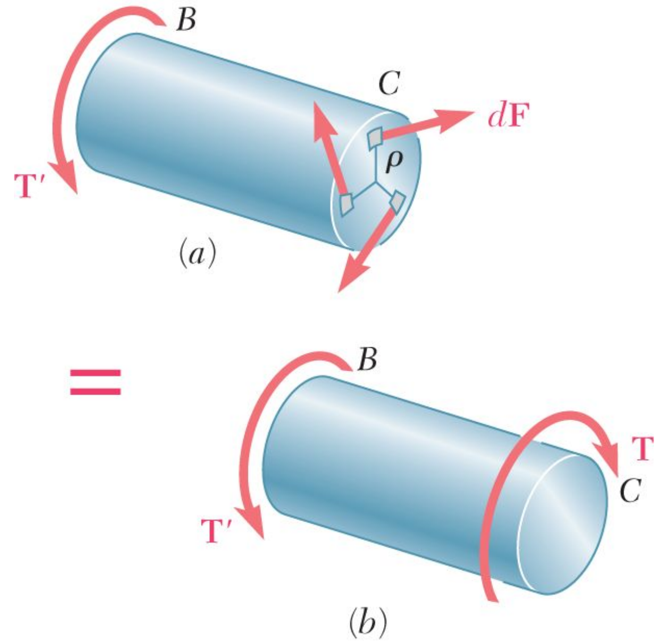
Torsion of shafts

- Refers to the twisting of a specimen when it is loaded by couples (or moments) that produce rotation about the longitudinal axis.
- Applications: aircraft engines, car transmissions, bicycles, etc.
- Units: Force × distance [lb.in] or [N.m]
- Torques are vector quantities and may be represented as follows:



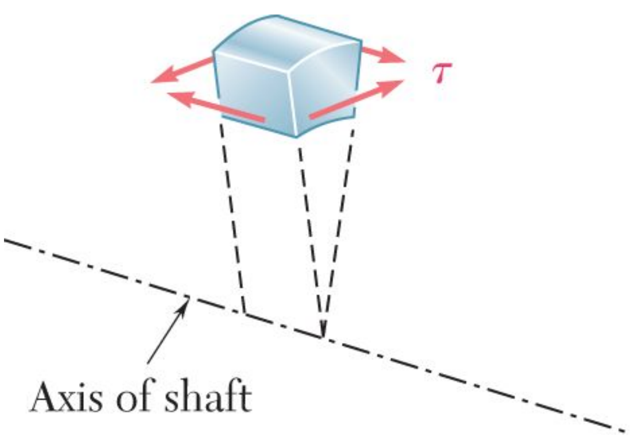
Equilibrium





$$T = \int \rho \, dF = \int \rho \, \tau \, dA$$

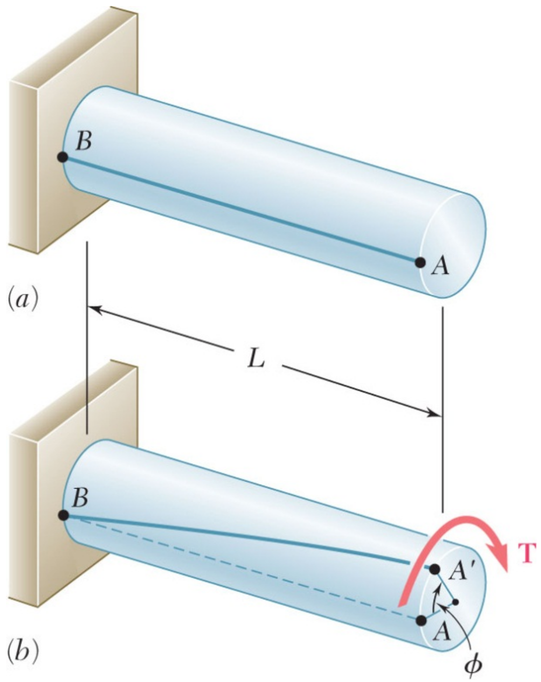
Although the net torque is known, the distribution of stresses is not.



- Stress distribution is statically indeterminate—must consider shaft deformations

- Shear stress cannot exist in one plane only—equilibrium requires the existence of shear stresses on the faces formed by the two planes containing the axis of the shaft.

Shaft deformations



Angle of twist: ϕ

Torque: T

Length: L

Diameter: d

From observation…

1) … the angle of twist of the shaft is:

A) proportional to the applied torque $\phi \propto T$

B) inversely proportional to the applied torque $\phi \propto \frac{1}{T}$

2) … the angle of twist of the shaft is:

A) proportional to the length $\phi \propto L$

B) inversely proportional to the length $\phi \propto \frac{1}{L}$

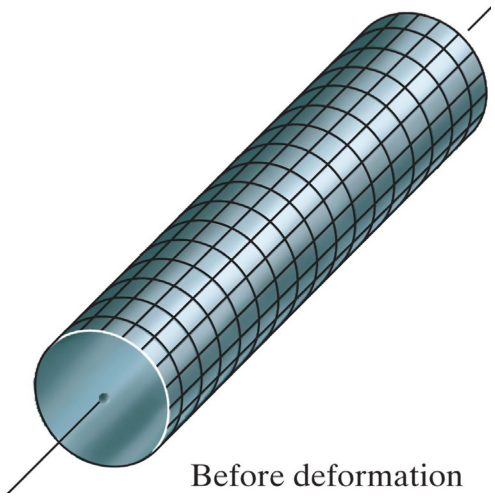
3) … the angle of twist of the shaft:

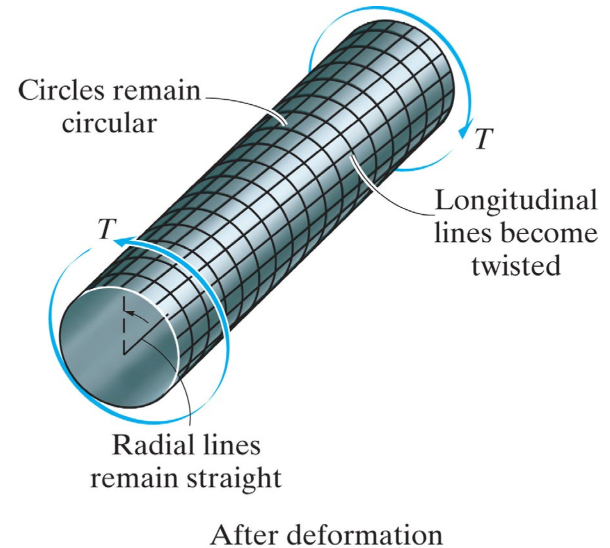
A) increases when the diameter of the shaft increases

B) decreases when the diameter of the shaft increases

Assumptions made about torsion deformation:

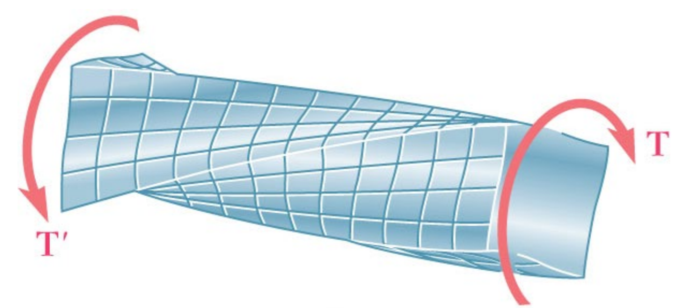
- For circular shafts (hollow and solid): cross-sections remain plane and undistorted due to axisymmetric geometry
  - i.e. while different cross sections have distinct angles of twist, each one of them rotates as a solid rigid slab
  - Longitudinal lines twist into a helix that intersects the circular cross sections at equal angles



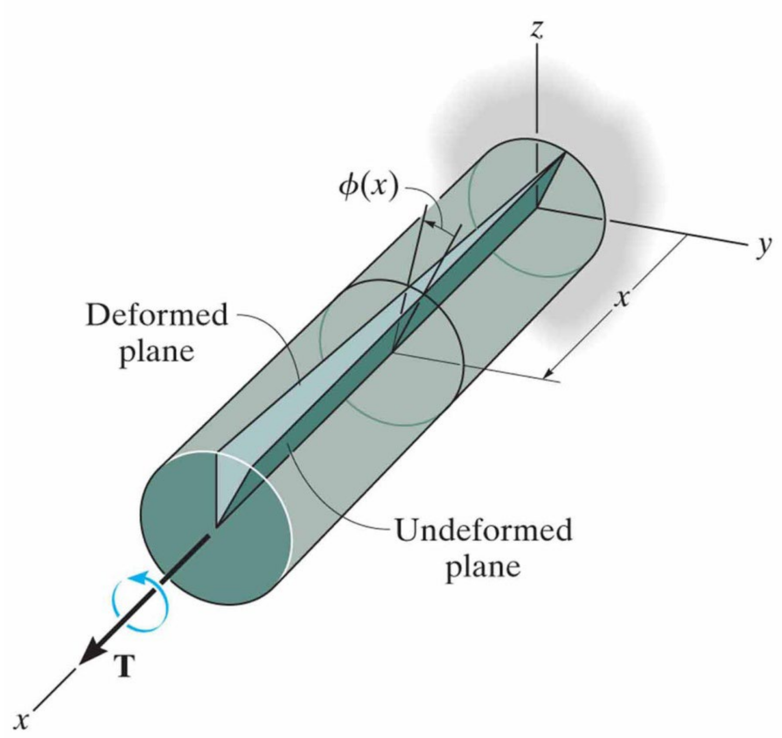


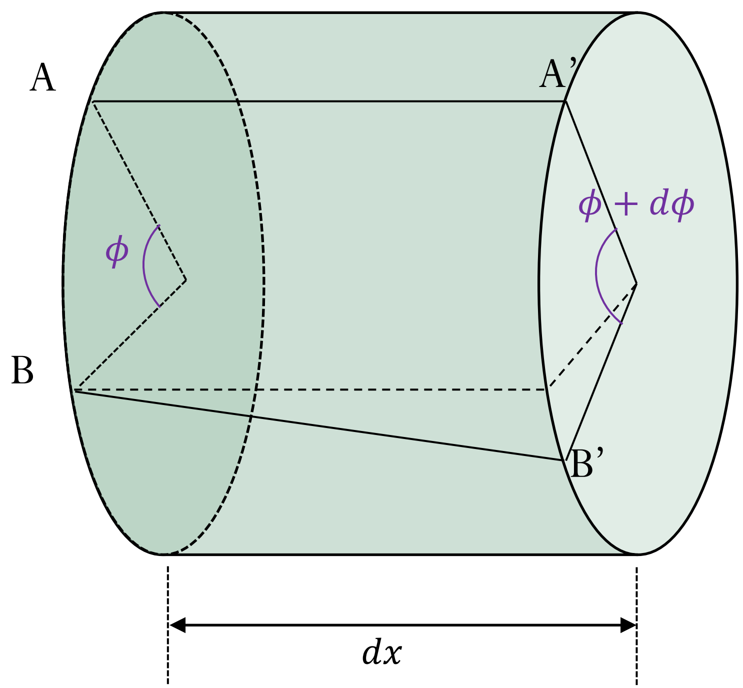
- Non-circular cross-sections warp and do not remain plane – we do not analyze these in TAM 251

- Linear and elastic deformation (small strains)

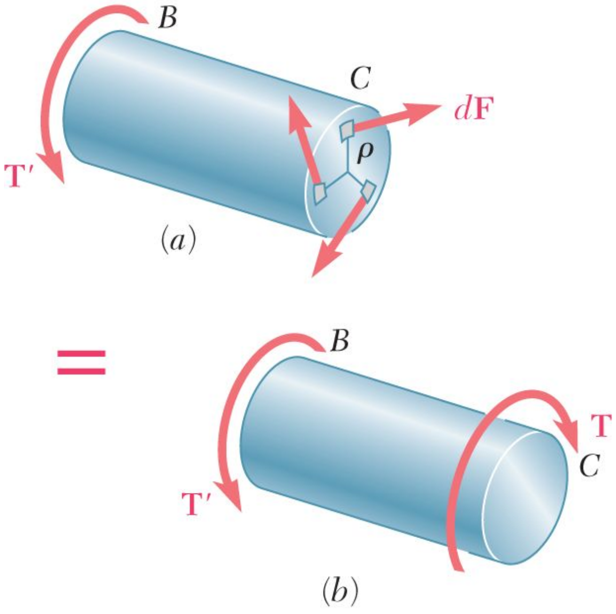


Shear strain – geometry of deformation





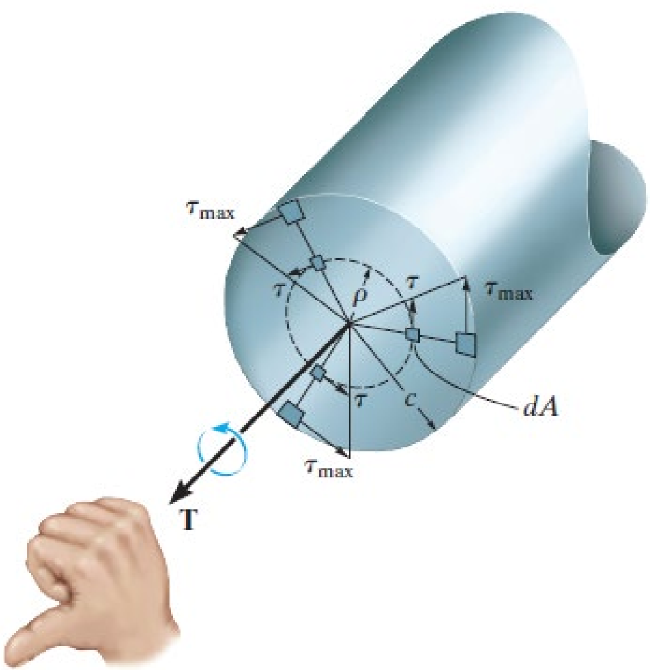
Shear stress distribution

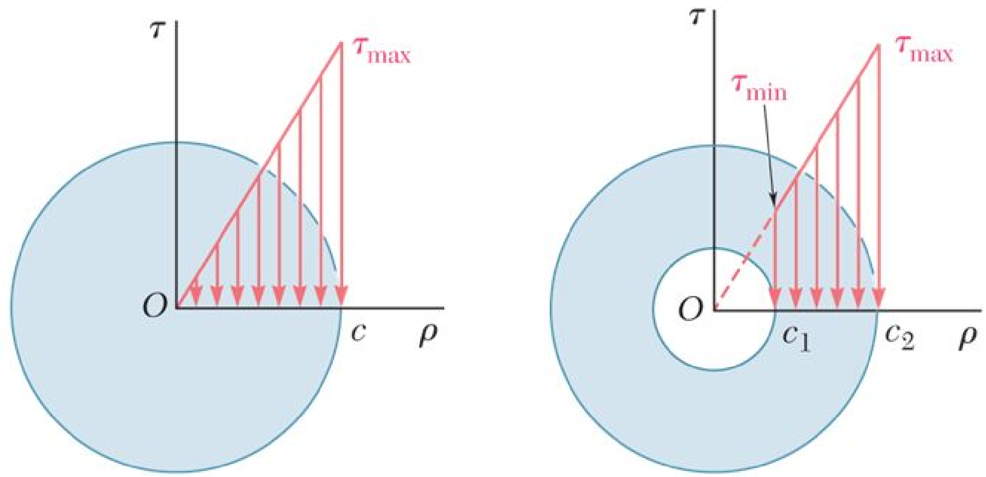


Shear stress in the elastic range

The shear stress varies linearly with the radial position in the section.

$$\tau = \frac{T\rho}{J}$$





(a) (b)

Polar moment of inertia

- Solid Shaft (radius R and diameter $D = 2R$):

$$J = \frac{\pi}{2} R^4 = \frac{\pi}{32} D^4$$

- Hollow Shaft (inner radius R_i and outer radius R_o)

$$J = \frac{\pi}{2}\left(R_o^4 - R_i^4\right) = \frac{\pi}{32}\left(D_o^4 - D_i^4\right)$$

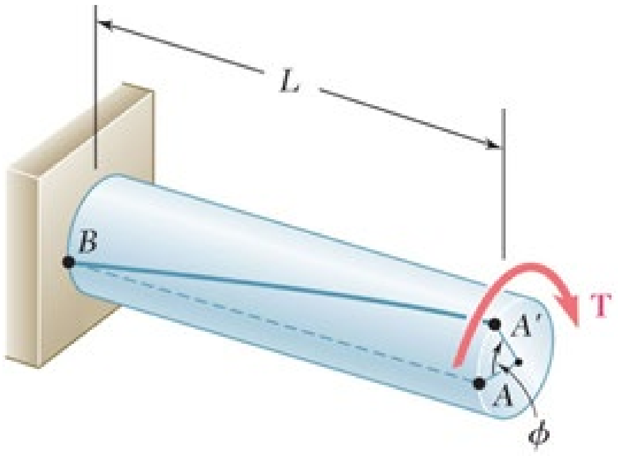
Angle of twist in the elastic range

$$\frac{d\phi}{dx} = \frac{T}{G\,J}$$

For constant torque and cross-sectional area:

$$\int_0^L \frac{d\phi}{dx}\,dx = \int_0^L \frac{T}{G\,J}\,dx$$

$$\phi \equiv \phi(L) - \phi(0) = \frac{T\,L}{G\,J}$$



$$\phi = \frac{T\,L}{G\,J}$$

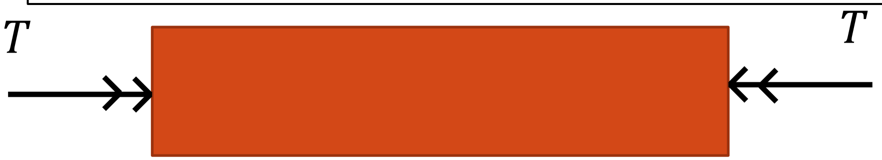
Torsional stiffness: $k_T = \dfrac{G\,J}{L}$

Torsional flexibility: $f_T = \dfrac{L}{G\,J}$

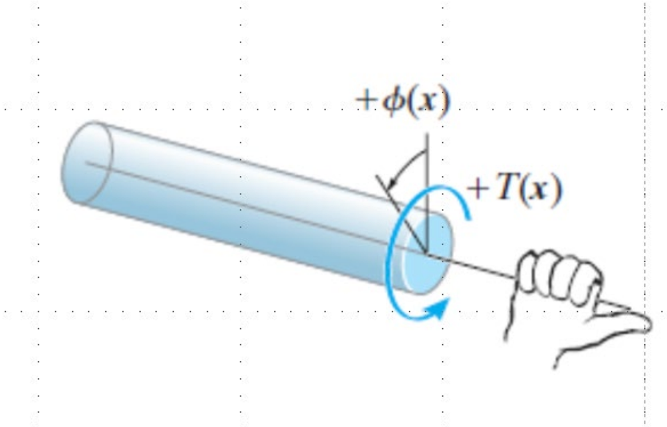
Sign conventions

Positive torque (points outward from faces)

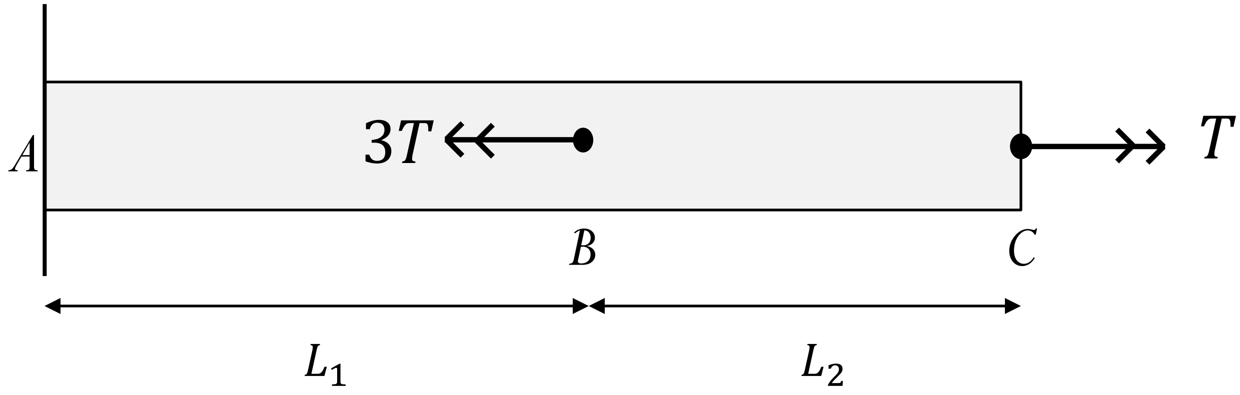
Negative torque (points inward towards faces)



Positive angle: CCW about torsion axis



Example



Find: ϕ_B, ϕ_C

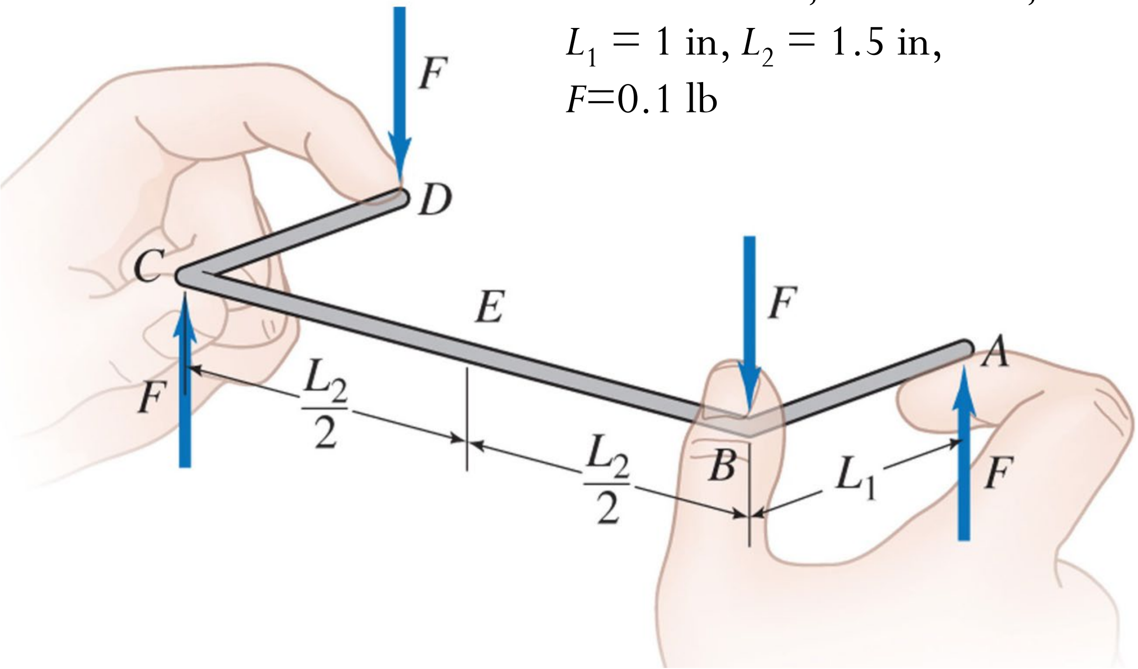
Given:

- shaft diameter D
- modulus G
- Lengths L_1 and L_2
- Applied torques at B and C

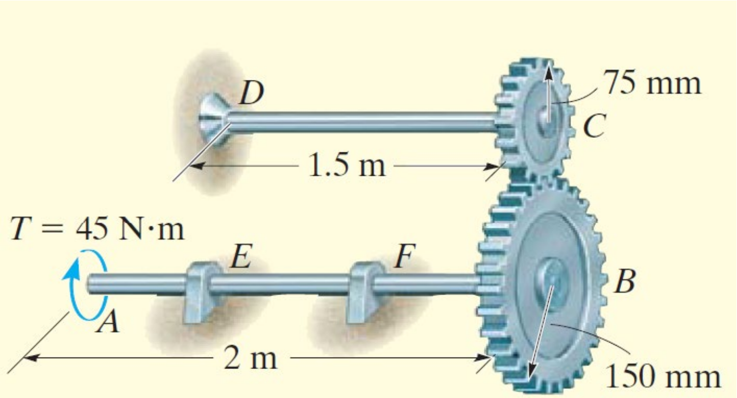
Example 1:

The bent steel wire is twisted by the four forces. Neglect bending of AB and CD due to the force F, and take B and C to have zero displacement. Assume the center plane E does not rotate, so A and D displace by equal amounts in the opposite directions. The wire has shear modulus G and diameter d. Determine the displacement of point D.

$G = 11*10^3$ ksi, $d = 0.04$ in,
$L_1 = 1$ in, $L_2 = 1.5$ in,
$F = 0.1$ lb



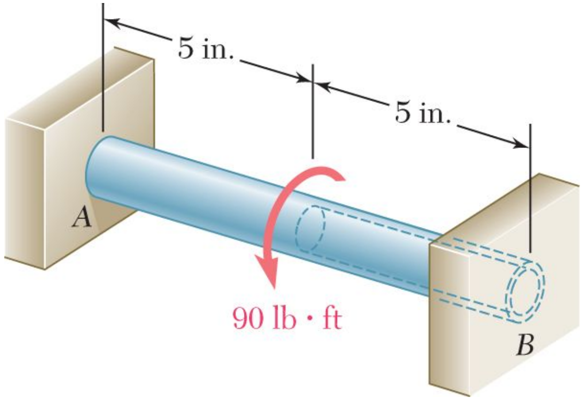
Gear systems with applied torque



(a)

Two solid steel shafts are connected through gears at points B and C. The top shaft is mounted to a fixed wall at point D. Determine the angle of twist at point A assuming a torque of $T = 45\ \text{N·m}$ is applied at A. Each shaft has a diameter of $d = 20\ \text{mm}$ and a shear modulus of $G = 80\ \text{GPa}$

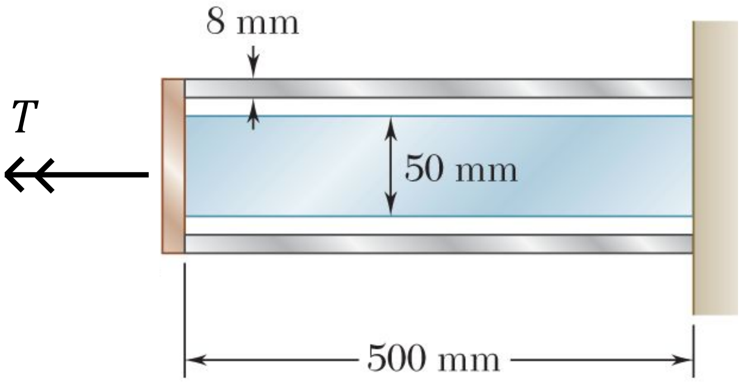
Statically Indeterminate Shafts



- Equilibrium:

- Geometry of deformation:

Statically Indeterminate Shafts

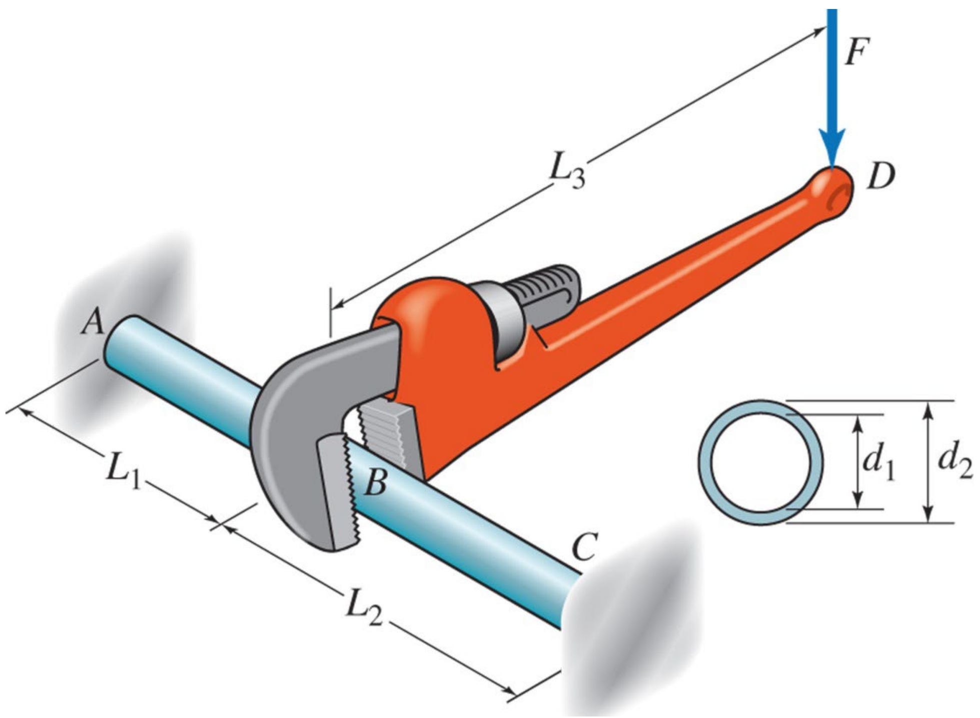


- Equilibrium:

- Geometry of deformation:

Example 3:

The plastic tube ($G = 1$ GPa) is being twisted with the wrench. Say the ends A and C are fixed, and the tube is supported against bending. A force $F = 40$ N is applied perpendicularly to the length of the wrench. Determine (a) the displacement of the end D of the wrench and (b) the maximum shear stress in the tube. Treat the wrench as rigid. ($L_1 = 100$ mm, $L_2 = 150$ mm, $L_3 = 250$ mm, $d_1 = 24$ mm, and $d_2 = 30$ mm.)



Example 4:

Given $T = 4\ kN \cdot m$, determine the maximum shear stress in shaft AB.