TAM251_Chapter6_Bending_prelecture_Johnson

Tuesday, July 6, 2021 1:54 PM

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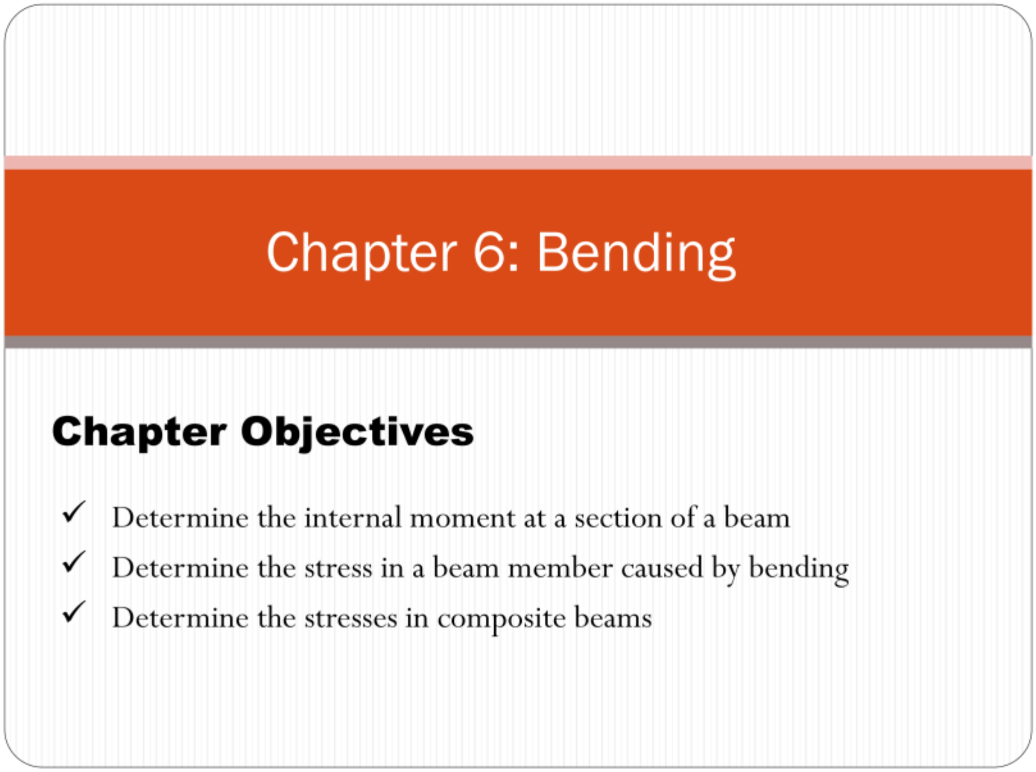
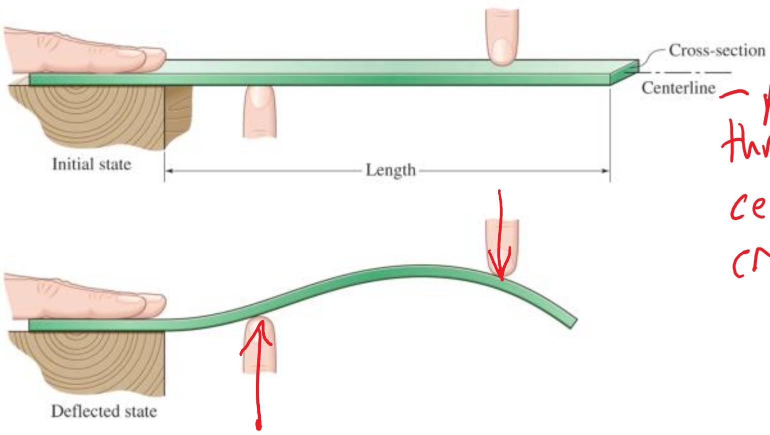


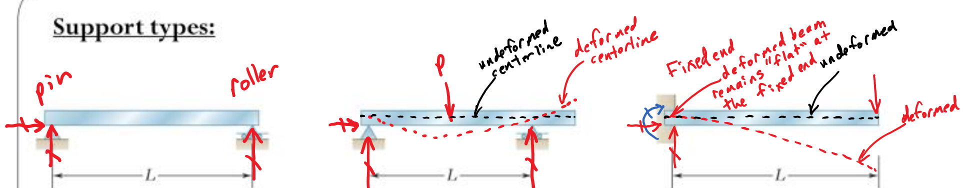
Bending analysis

This wood ruler is held flat against the table at the left, and fingers are poised to press against it. When the fingers apply forces, the ruler deflects, primarily up or down. Whenever a part deforms in this way, we say that it acts like a "beam." In this chapter, we learn to determine the stresses produced by the forces and how they depend on the beam cross-section, length, and material properties.



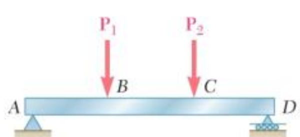
Centerline – passes through geometric centroid of the cross-section

Support types:

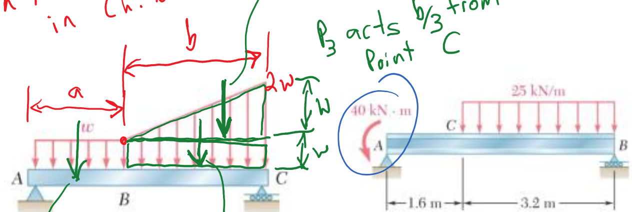


(*a*) Simply supported beam (*b*) Overhanging beam (*c*) Cantilever beam

Load types: (transverse only for now) we won't use $\sigma = F/A$ in Ch. 6



Concentrated loads
or "point loads"



Distributed loads

$P_1 = w \cdot a$

$[w] = \frac{\text{force}}{\text{length}}$

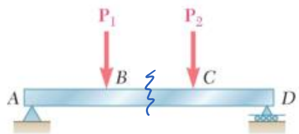
$P_2 = w \cdot b$ acts halfway between B & C

$P_3 = \frac{1}{2} \cdot b \cdot w$ (b = length, w = force/length)

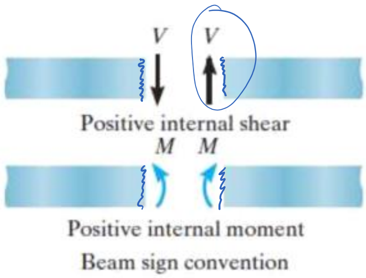
P_3 acts b/3 from point C

Concentrated moments
"point" moments
their sign follows the right-hand rule
– positive if thumb points out of the screen (or page)

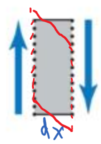
Sign conventions:



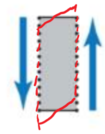
How do we define whether the internal shear force and bending moment are positive or negative?



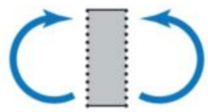
If $V > 0$, we mean



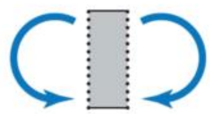
If $V < 0$, we mean



If $M > 0$, we mean

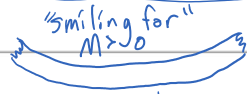


If $M < 0$, we mean



"smiling for" $M>0$

"frowning" for $M<0$

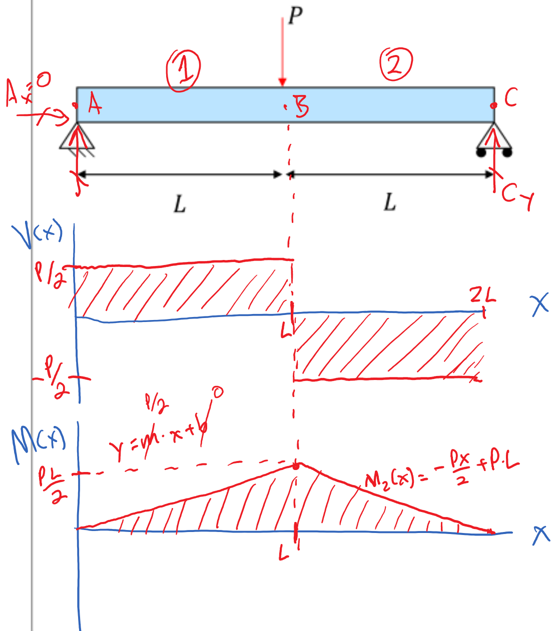


concave up beam deflection curve

concave down beam deflection curve

Shear and moment diagrams

Given:	Find:
$P = 2$ kN $L = 1$ m	$V(x), M(x)$, Shear-Bending moment diagram along the beam axis



1. Find reactions

$\Sigma F_x = 0 \Rightarrow A_x = 0$

$\Sigma M_A = 0 \Rightarrow -P \cdot L + C_y \cdot 2L = 0$

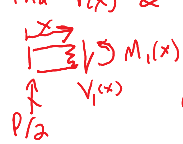
$\Rightarrow C_y = P/2$

$\Sigma F_y = 0$

$\Rightarrow A_y + C_y - P = 0$

$\therefore A_y = P/2$

2. Find $V(x)$ & $M(x)$ in section ① $(0 \le x < L)$



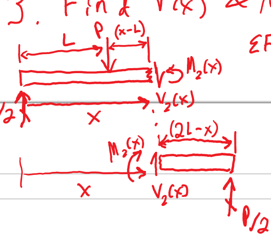
$\Sigma F_y = 0 \Rightarrow \frac{P}{2} - V_1(x) = 0$

$\Rightarrow V_1(x) = P/2$

$(\Sigma M)_x = 0 \Rightarrow M_1(x) - \frac{P}{2} \cdot x = 0$

$M_1(x) = \frac{1}{2} \cdot P \cdot x$

3. Find $V(x)$ & $M(x)$ in section ② $(L < x \le 2L)$



$\Sigma F_y = 0 \Rightarrow \frac{P}{2} - P - V_2(x) = 0$

$\Rightarrow V_2(x) = -\frac{P}{2}$

$(\Sigma M)_x = 0$

$\Rightarrow M_2(x) + P(x - L) - \frac{P}{2} \cdot x = 0$

$M_2(x) = \left(-\frac{P}{2}\right)x + (P \cdot L)$

$= P \cdot \left(L - \frac{x}{2}\right)$

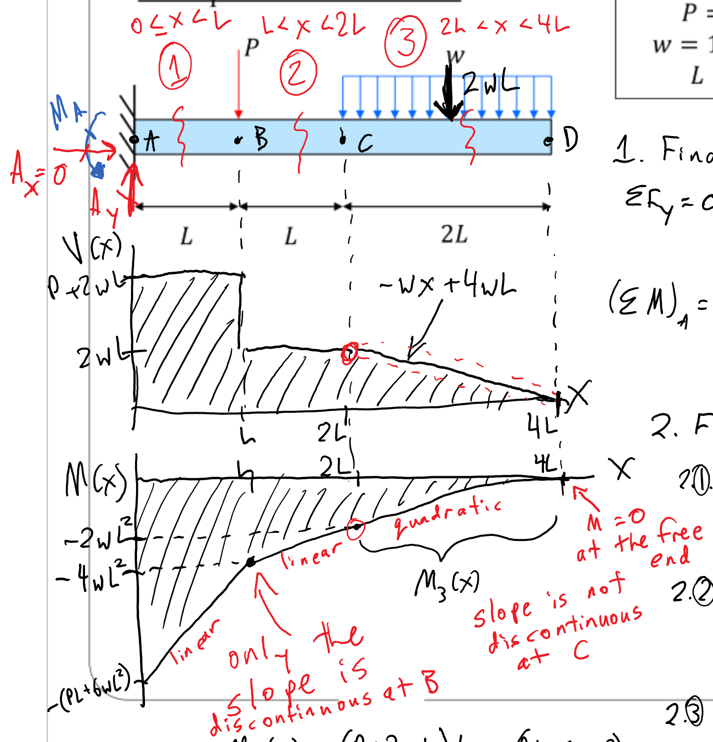
$y = m \cdot x + b$, $m \to -P/2$, $b \to P \cdot L$

Given:	Find:

Example: cantilever beam

Given:
$P = 4$ kN
$w = 1.5$ kN/m
$L = 1$ m

Find:
$V(x), M(x)$, Shear-Bending moment diagram along the beam axis



1. Find reactions

$\Sigma F_y = 0 \Rightarrow A_y - P - w\cdot 2L = 0$

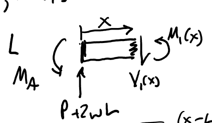
$A_y = P + 2wL$

$(\Sigma M)_A = 0 \Rightarrow M_A - P\cdot L - (2wL)(3L) = 0$ (force, moment arm)

$M_A = PL + 6wL^2$

2. Find $V(x)$, $M(x)$ in sections ①, ② & ③

2.① $0 \le x < L$



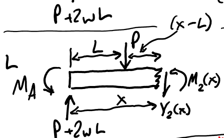
$\Sigma F_y = 0 \Rightarrow P + 2wL - V_1(x) = 0$

$V_1(x) = P + 2wL$

$(\Sigma M)_x = 0 \Rightarrow M_1(x) - (P+2wL)x + (PL + 6wL^2) = 0$

$M_1(x) = (P+2wL)\cdot x - (PL+6wL^2)$

2.② $L < x < 2L$



$\Sigma F_y = 0 \Rightarrow P + 2wL - P - V_2(x) = 0$

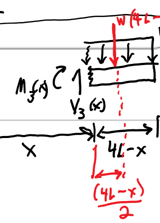
$\Rightarrow V_2(x) = 2wL$

$(\Sigma M)_x = 0 \Rightarrow M_2(x) + P\cdot(x-L) - (P+2wL)\cdot x + (PL+6wL^2) = 0$ (x-terms)

$M_2(x) = (P + 2wL - P)x - (PL + 6wL^2) + P\cdot L$

$M_2(x) = 2wL\cdot x - 6wL^2$

2.③ $2L < x < 4L$



$\Sigma F_y = 0 \Rightarrow V_3(x) - w\cdot(4L - x) = 0$

$V_3(x) = (-w)x + (4wL) \quad 2L < x \le 4L$

$(\Sigma M)_x = 0 \Rightarrow -M_3(x) - w\cdot(4L-x)\dfrac{(4L-x)}{2} = 0$ (force, arm)

$M_3(x) = -\frac{w}{2}(4L-x)^2$

$= -\frac{w}{2}(16L^2 - 8Lx + x^2)$

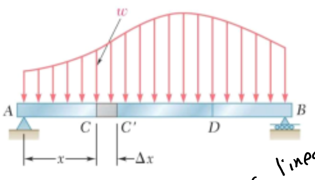
$M_1(L) = (P + 2wL)L - (PL + 6wL^2)$

$= -4wL^2$

$M_2(2L) = 2wL(2L) - 6wL^2$

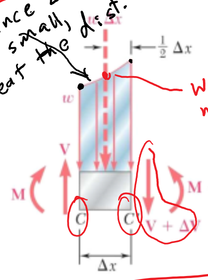
$= -2wL^2$

Relations Among Load, Shear and Bending Moments



(a)

Since Δx is small, treat the dist. load as linear



w in the middle

(b)

Relationship between load and shear:

$$\Sigma F_y = 0 \Rightarrow V - (V + \Delta V) - w \cdot \Delta x = 0$$

$$-\Delta V - w \cdot \Delta x$$

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = -w$$

$$\boxed{\frac{dV}{dx} = -w}$$

Integrate to get

$$\int_{V_C}^{V_{C'}} dV = \boxed{V_{C'} - V_C = -\int_{x_C}^{x_{C'}} w \cdot dx}$$

Relationship between shear and bending moment:

$$(\Sigma M)_{C'} = 0 \Rightarrow M + \Delta M - M + w \Delta x \cdot \frac{\Delta x}{2} - V \cdot \Delta x = 0$$

$$\Delta M + \frac{1}{2} w (\Delta x)^2 - V \cdot \Delta x = 0$$

Divide by Δx:

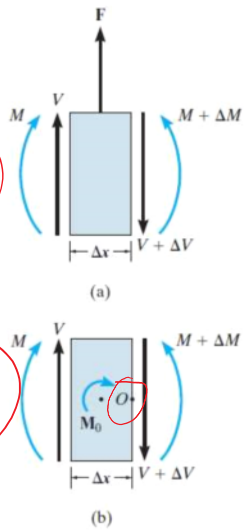
$$\frac{\Delta M}{\Delta x} = V - \frac{1}{2} w \cdot \Delta x$$

take $\lim \Delta x \to 0 \Rightarrow$ $\boxed{\frac{dM}{dx} = V}$

$$\boxed{M_{C'} - M_C = \int_{x_C}^{x_{C'}} V(x) \cdot dx}$$

Wherever there is an external concentrated force, or a concentrated moment, there will be a change (jump) in shear or moment respectively.

Concentrated force F acting upwards:



Concentrated Moment M_0 acting clockwise

Fig. 6–10

$\Sigma F_y = 0 \quad V - (V + \Delta V) + F = 0$

$\Delta V = F$ jump in V(x) equals the value of the point load

$(\Sigma M)_O = 0$

$\Rightarrow M + \Delta M - M - M_0 - V \cdot \Delta x = 0$

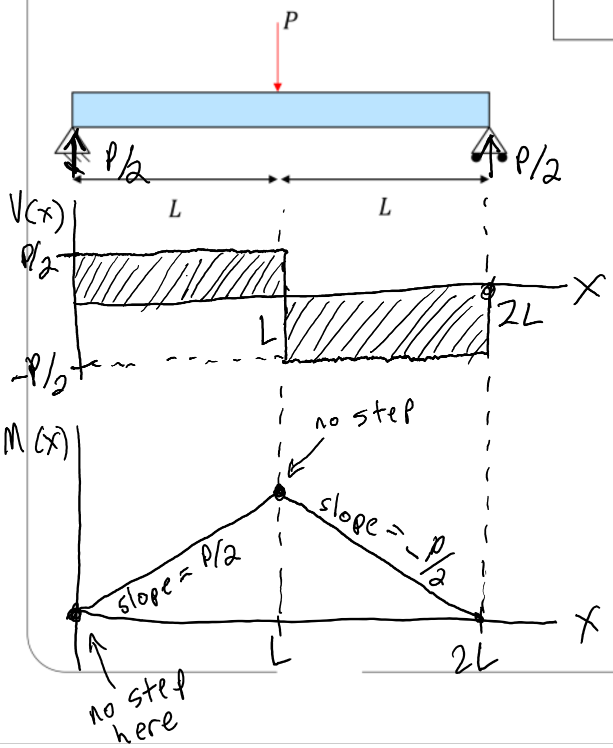
$\Delta M = M_0 + V \cdot \Delta x$

say Δx small: $\Delta M = M_0$ (here $M_0 < 0$)

Shear and moment diagrams

Given:
$P = 2$ kN
$L = 1$ m

Use the graphical method the sketch diagrams for V(x) and M(x)



$\Delta V =$ value of point load (move left-to-right) (direction of increasing x)

$\frac{dM}{dx} = V(x) \qquad \Delta M = -$ pt. moment (but no pt. moments in this beam!)

In region ①

$M(x) - M(0) = \int_0^x V(x')\,dx'$

$M(x) = \int_0^x P/2\,dx' = \frac{P}{2}(x - 0) = \frac{Px}{2}$

here, we know $M_2(L) = M_1(L)$

here, we know $M_2(L) = M_1(L)$

In region ②

$$M_2(x) - M_2(L) = \int_L^x V_2(x')\,dx'$$

get $M(L)$ from $M_1(x)$

$$M_1(x) = \frac{P \cdot x}{2} \Rightarrow M_1(L) = \frac{PL}{2}$$

$$M_2(x) - \frac{PL}{2} = \int_L^x -\frac{P}{2} \cdot dx'$$

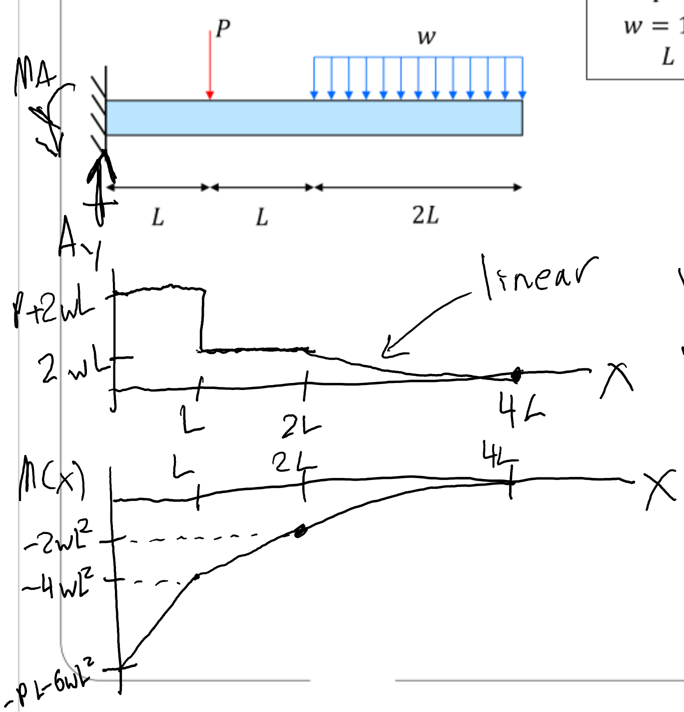
$$M_2(x) = \frac{PL}{2} - \frac{P}{2}(x-L)$$

$$= -\frac{P}{2}(x-L)$$

Example: cantilever beam

Given:
$P = 4$ kN
$w = 1.5$ kN/m
$L = 1$ m

Use the graphical method the sketch diagrams for V(x) and M(x)



$V_1(x) = P + 2wL$ in $0 \le x \le L$

$V_2(x) = 2wL$ in $L < x < 2L$

$V_3(x) = -wx + 4wL$ in $2L < x < 4L$

$$M_1(x) - M_1(0) = \int_0^x V(x')\,dx'$$

$M_1(0)$ follows the pt. moment rule

$$M_A = PL + 6wL^2$$

$$\Rightarrow M_1(0) = -M_A = -(PL + 6wL^2)$$

$$M_1(x) = M_1(0) + \int_0^x (P + 2wL)\,dx'$$

$$= -(PL + 6wL^2) + (P + 2wL)x$$

$$M_1(L) = -4wL^2$$

$$M_2(x) - M_2(L) = \int_L^x V_2(x')\cdot dx'$$

$$M_2(L) = M_1(L) = -4wL^2$$

$$\Rightarrow M_2(x) = -4wL^2 + \int_L^x 2wL\cdot dx'$$

$$= -4wL^2 + 2wL(x-L)$$

$$= 2wLx - 6wL^2$$

$$M_3(2L) = M_2(2L) = -2wL^2$$

$$V_3(x) - V_3(2L) = -\int_{2L}^x w(x')dx'$$

Note $V_3(2L) = V_2(2L) = 2wL \Rightarrow V_3(x) = 2wL - \int_{2L}^x (w)\,dx'$

$$= 2wL - w(x-2L)$$

$$= -w\cdot x + 4wL$$

$$V_3(2L) = -w(2L) + 4wL$$

$$= 2wL$$

$$M_2(L) = 2wL^2 - 6wL^2 = -4wL^2$$

$$M_2(2L) = 2wL(2L) - 6wL^2$$

$$= 4wL^2 - 6wL^2 = -2wL^2$$

$$M_3(x) - M_3(2L) = \int_{2L}^x V_3(x')\cdot dx'$$

$$\underline{M_3(x)} = \underline{M_3(2L)} + \int_{2L}^x (-wx' + 4wL)\,dx'$$

$$= \underline{-2wL^2} + (-w)\frac{(x')^2}{2}\Big|_{2L}^x + 4wL\cdot(\underline{x-2L})$$

$$= -\underline{2wL^2} - \frac{w}{2}(x^2 - 4L^2) + \underline{4wLx} - \underline{8wL^2}$$

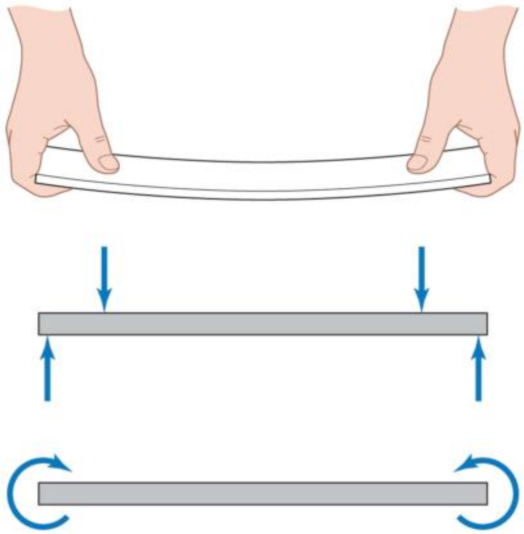
$$= -\frac{w}{2}x^2 + 4wLx - 8wL^2$$

$$M_3(4L) = -\frac{w}{2}(4L)^2 + 4wL(4L) - 8wL^2$$

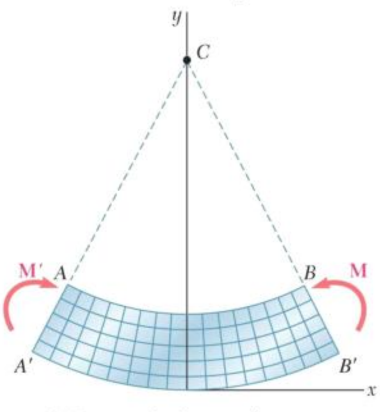
$$= -8wL^2 + 16wL^2 - 8wL^2 = 0$$

Pure bending

Take a flexible strip, such as a thin ruler, and apply equal forces with your fingers as shown. Each hand applies a couple or moment (equal and opposite forces a distance apart). The couples of the two hands must be equal and opposite. Between the thumbs, the strip has deformed into a circular arc. For the loading shown here, just as the deformation is uniform, so the internal bending **moment is uniform**, equal to the moment applied by each hand.



Geometry of deformation



(*a*) Longitudinal, vertical section (plane of symmetry)

We assume that "plane sections remain plane" → All faces of "grid elements" remain at 90^o to each other, hence

$$\gamma_{xy} = \gamma_{xz} = 0$$

Therefore,

$$\tau_{xy} = \tau_{xz} = 0$$

No external loads on y or z surfaces:

$$\sigma_y = \sigma_z = \tau_{yz} = 0$$

Thus, at any point of a slender member in pure bending, we have a **state of uniaxial stress**, since σ_x is the only non-zero stress component

For positive moment, $M > 0$ (as shown in diagram):

Segment *AB* decreases in length → $\sigma_x < 0$ and $\epsilon_x < 0$

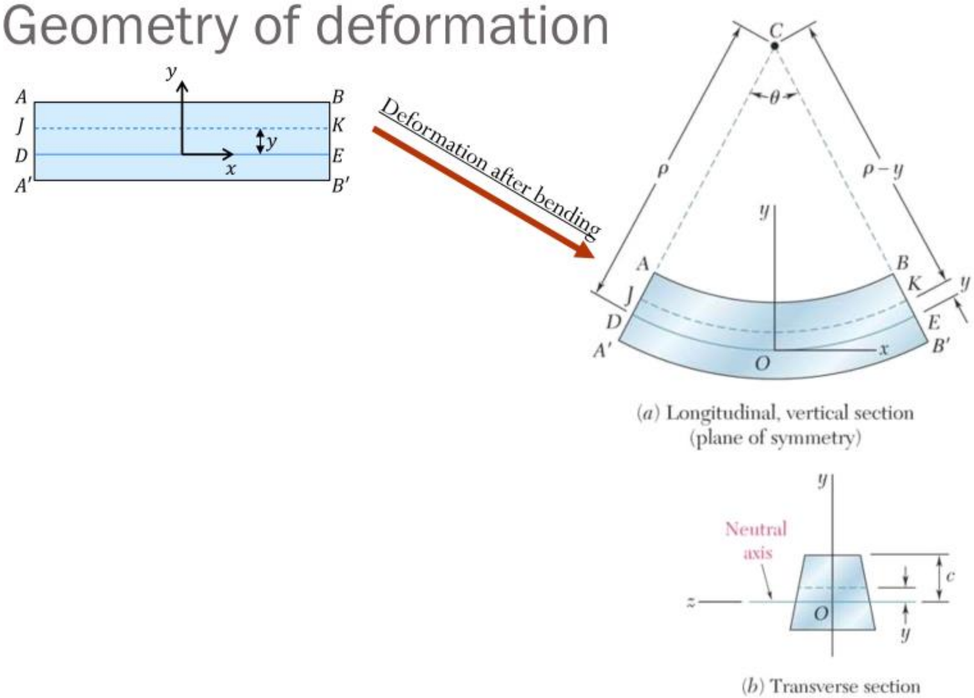
Segment *A'B'* increases in length → $\sigma_x > 0$ and $\epsilon_x > 0$

→ Hence there must exist a surface parallel to the upper and lower where

$$\sigma_x = 0 \text{ and } \epsilon_x = 0$$

This surface is called **NEUTRAL AXIS**

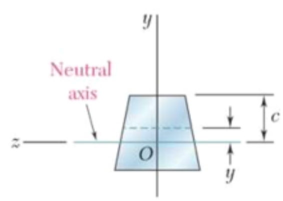
Geometry of deformation



Constitutive and Force Equilibrium

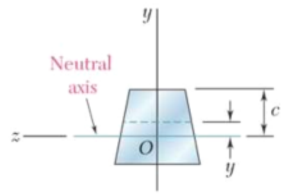
Constitutive relationship: $\epsilon_x = \frac{-y}{\rho}$ ➡ $\sigma_x = E\epsilon_x = -\frac{Ey}{\rho}$

Force equilibrium:



(b) Transverse section

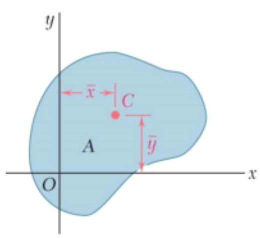
Moment Equilibrium



(b) Transverse section

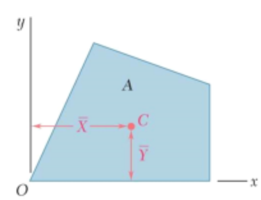
Centroid of an area

- The centroid of the area A is defined as the point C of coordinates $\bar{x}$ and $\bar{y}$, which satisfies the relation



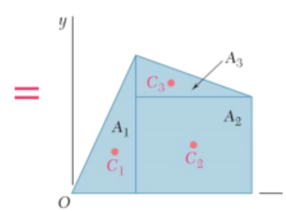
$$\int_A x\, dA = A\,\bar{x}$$

$$\int_A y\, dA = A\,\bar{y}$$

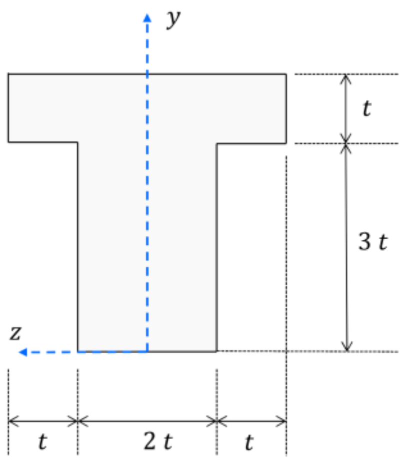


- In the case of a composite area, we divide the area A into parts A_1, A_2, A_3

$$A_{total}\bar{x} = \sum_i A_i\bar{x}_i \qquad A_{total}\bar{y} = \sum_i A_i\bar{y}_i$$

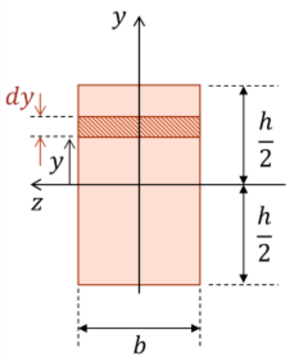


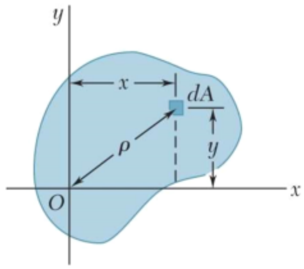
Example: Find the centroid position in the yz coordinate system shown for $t = 20\text{cm}$



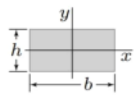
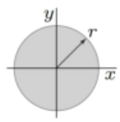
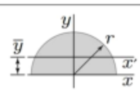
Second moment of area

- The 2nd moment of the area A with respect to the x-axis is given by $I_x = \int_A y^2 \, dA$
- The 2nd moment of the area A with respect to the y-axis is given by $I_y = \int_A x^2 \, dA$
- Example: 2nd moment of area for a rectangular cross section:





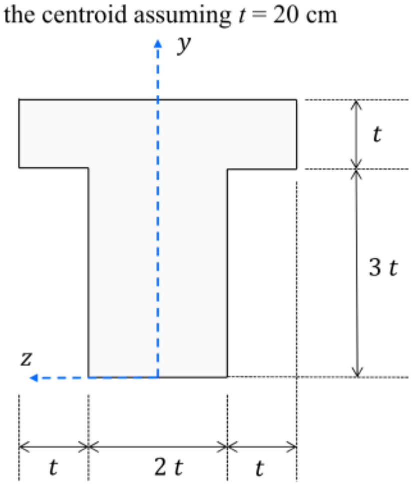
Centroids and area moments of area: Formula sheet

Moments and Geometric Centroids					
		$Q = \overline{y} A$	$I_x = \int_A y^2 \, dA$	$J_o = \int_A \rho^2 \, dA$	$\overline{y} = \frac{1}{A} \int_A y \, dA$
Rectangle			$I_x = \frac{1}{12} b h^3$		
Circle			$I_x = \frac{\pi}{4} r^4$	$J_z = \frac{\pi}{2} r^4$	
Semicircle			$I_{x'} = \left(\frac{\pi}{8} - \frac{8}{9\pi} \right) r^4$		$\overline{y} = \frac{4r}{3\pi}$
Parallel Axis Theorem		$I_c = I_{c'} + A d_{cc'}^2$			

Parallel-axis theorem: the 2nd moment of area about an axis through C parallel to the axis through the centroid C' is given by

$$I_C = I_{C'} + A \, d_{CC'}^2$$

Example: Find the 2nd moment of area about the horizontal axis passing through the centroid assuming $t = 20$ cm

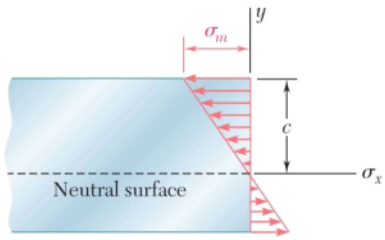


Bending stress formula

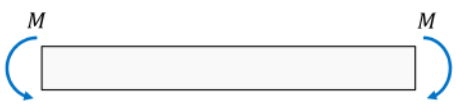
$$\sigma_x(x, y) = -\frac{M(x)\,y}{I_z(x)}$$

- The maximum magnitude occurs the furthest distance away from the neutral axis. If we denote this maximum distance "c", consistent with the diagram below, then we can write

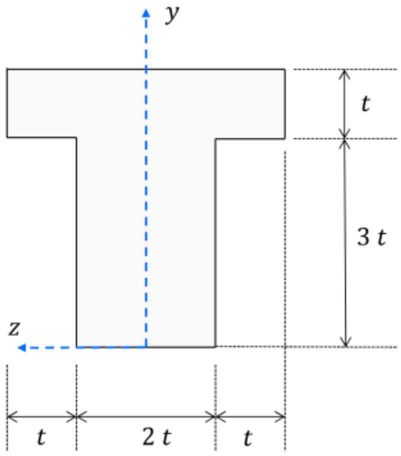
$$\sigma_m = \frac{|M|c}{I_z}$$



Bending stress sign



Example: Find the maximum tensile and compressive stresses in this beam subjected to moment M_z = 100 N-m with the moment vector pointing in the direction of the z-axis. Again take t = 20 cm.



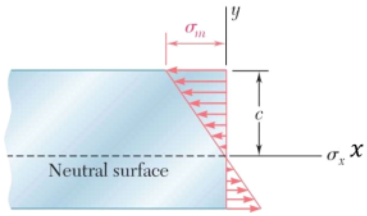
Why I-beams?

http://studio-tm.com/constructionblog/wp-content/uploads/2011/12/steel-i-beam-cantilevered-over-concrete-wall.jpg

Summary of bending in beams

- Maximum stress due to bending

$$\sigma = \frac{Mc}{I}$$



- Bending stress is zero at the neutral axis and ramps up linearly with distance away from the neutral axis

- I is the 2nd moment of area about the neutral axis of the cross section
    - Be sure to find the cross-section's centroid and evaluate I about an axis passing through the centroid, using the parallel axis theorem if needed

$$I_C = I_{C'} + A\, d^2_{CC'}$$

- To determine stress sign, look at the internal bending moment direction:
    - Side that moment curls **towards** is in **compression**
    - Side that moment curls **away from** is in **tension**