Chapter 9: Stress Transformation

Chapter Objectives

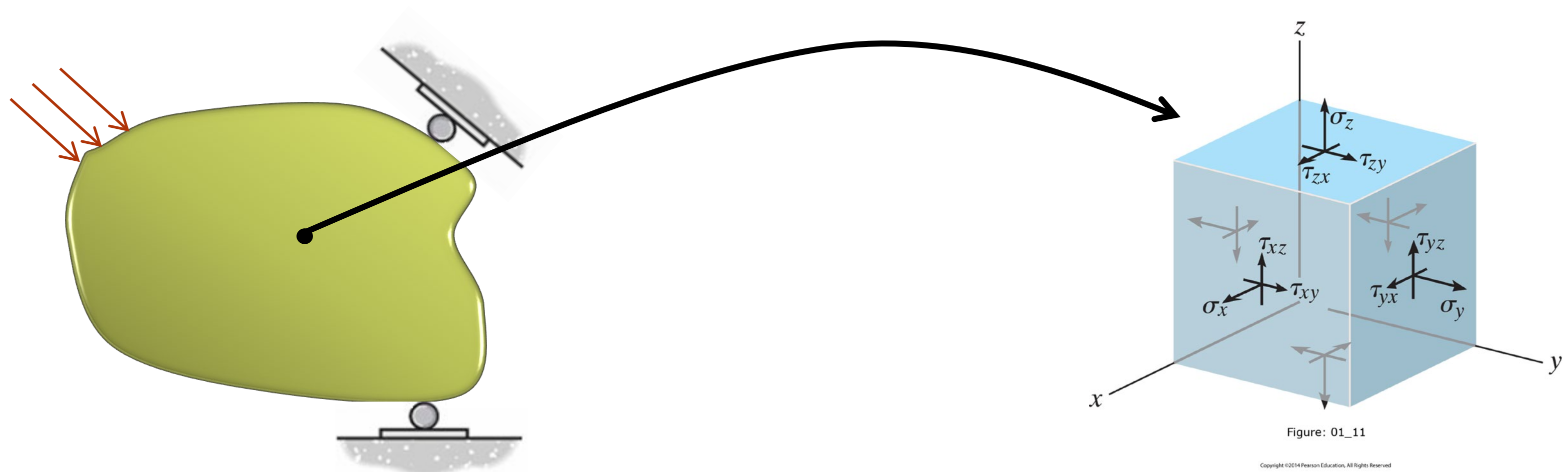
✓ Navigate between rectilinear coordinate systems for stress components
✓Determine principal stresses and maximum in-plane shear stress
✓Determine the absolute maximum shear stress in 2D and 3D cases

General stress state

The general state of stress at a point is characterized by

- three independent normal stress components $\sigma_x, \sigma_y,$ and σ_z
- three independent shear stress components $\tau_{xy}, \tau_{yz},$ and τ_{xz}

At a given point, we can draw a stress element that shows the normal and shear stresses acting on the faces of a small (infinitesimal) cube of material surrounding the point of interest



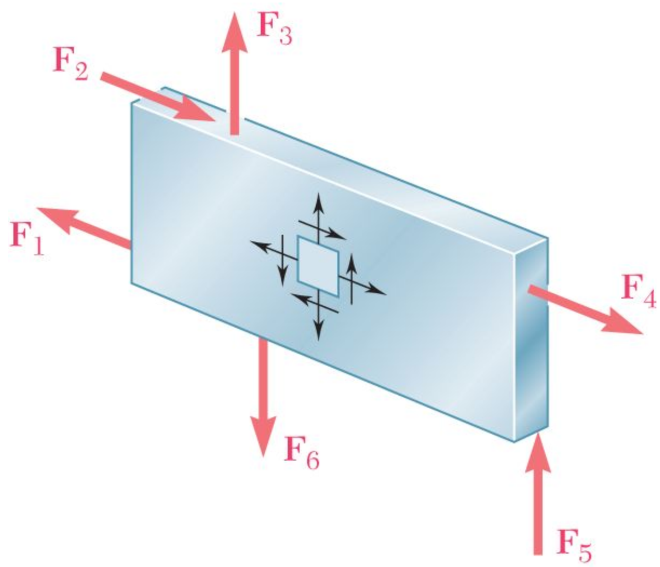
Plane Stress

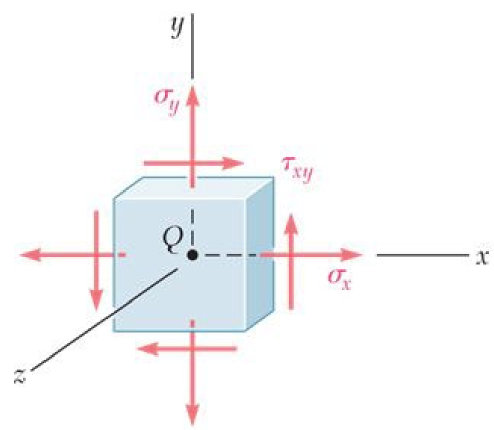
- Often, a loading situation involves only loads and constraints acting applied within a two-dimensional plane (e.g. the xy plane). In this case, any stresses acting in the third plane (z in this case) are equal to zero:

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

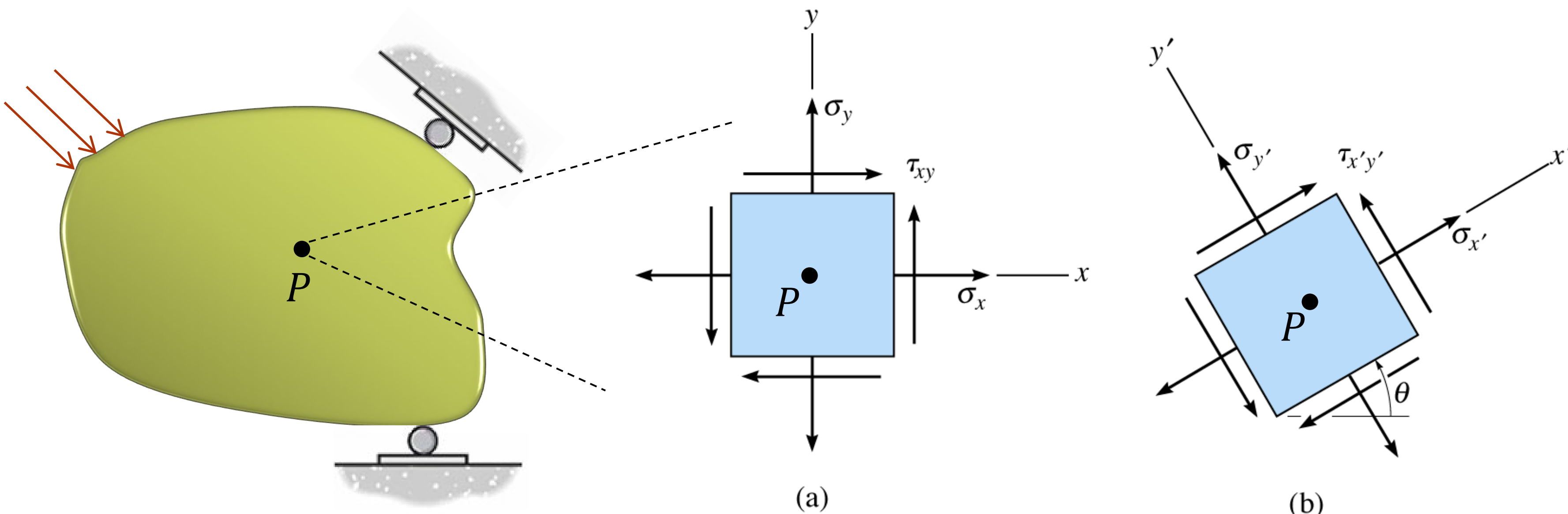
- Example:

Thin plates subject to forces acting in the mid-plane of the plate





Plane Stress Transformation

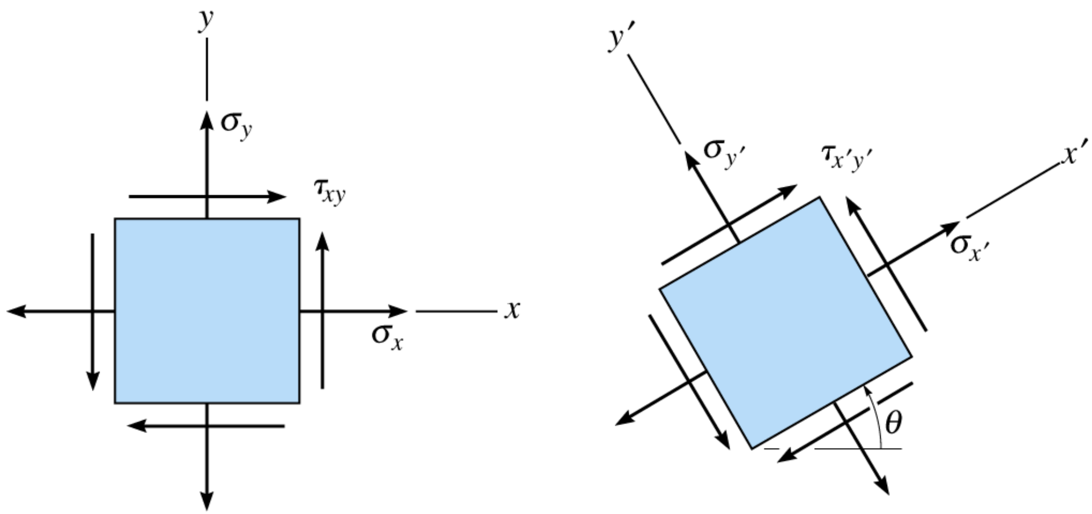


The stress tensor gives the normal and shear stresses acting on the faces of a cube (square in 2D) whose faces align with a particular coordinate system.

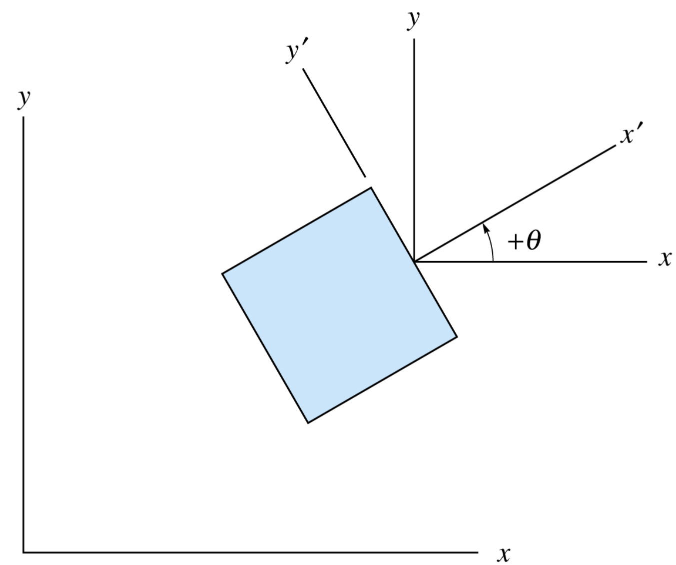
But, the choice of coordinate system is arbitrary. We are free to express the normal and shear stresses on any face we wish, not just faces aligned with a particular coordinate system.

Stress transformation equations give us a formula/methodology for taking known normal and shear stresses acting on faces in one coordinate system (e.g. x-y above) and converting them to normal and shear stresses on faces aligned with some other coordinate system (e.g. x'-y' above)

Plane Stress Transformation

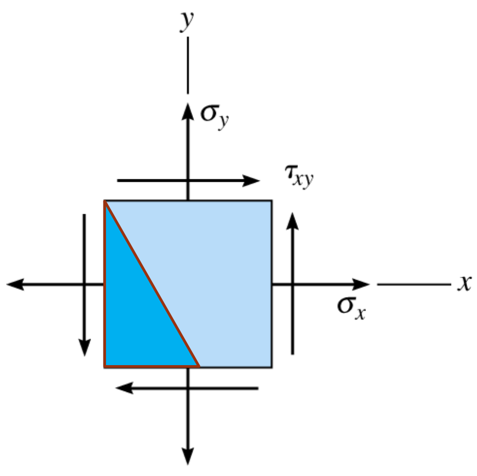


(a) (b)

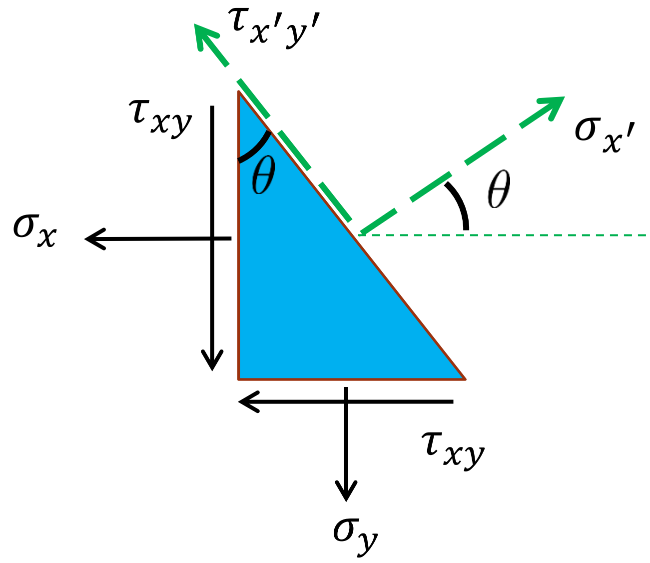


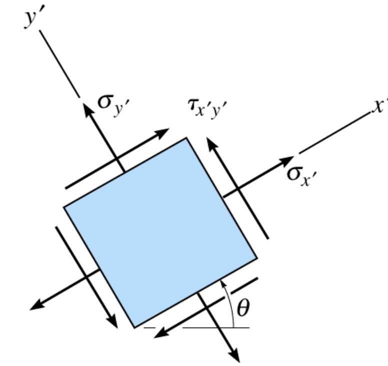
- Sign convention:
  - Both the x-y and x’-y’ system follow the right-hand rule
  - The orientation of an inclined plane (on which the normal and shear stress components are to be determined) will be defined using the angle θ. The angle θ is measured from the positive x to the positive x’-axis. It is positive if it follows the curl of the right-hand fingers.

For two-dimensional problems:



(a)





(b)

We use the following trigonometric relations…

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \sin(2\theta) = 2\sin\theta\,\cos\theta$$

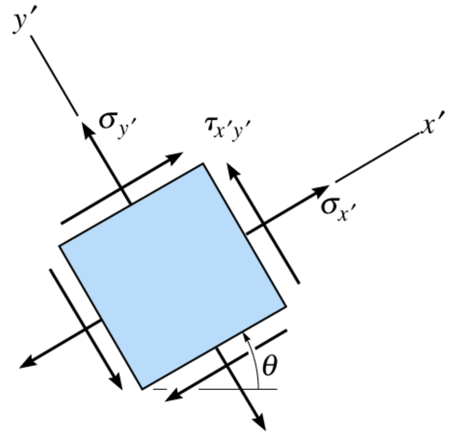
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

… to get

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

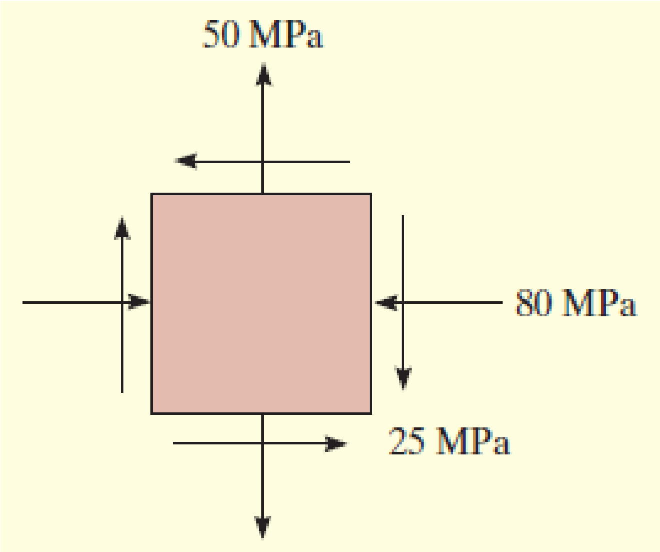
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$



(b)

Note that: $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$

Example 1: The state of plane stress at a point is represented by the element shown in the figure below. Determine the state of stress at the point on another element oriented 30° clockwise from the position shown.



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

Principal Stresses

At what angle is the normal stress $\sigma_{x'}$ maximized/minimized? Start from:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

There are two roots (that we care about):

$$\theta_{p1} \text{ and } \theta_{p2} = \theta_{p1} + 90^o$$

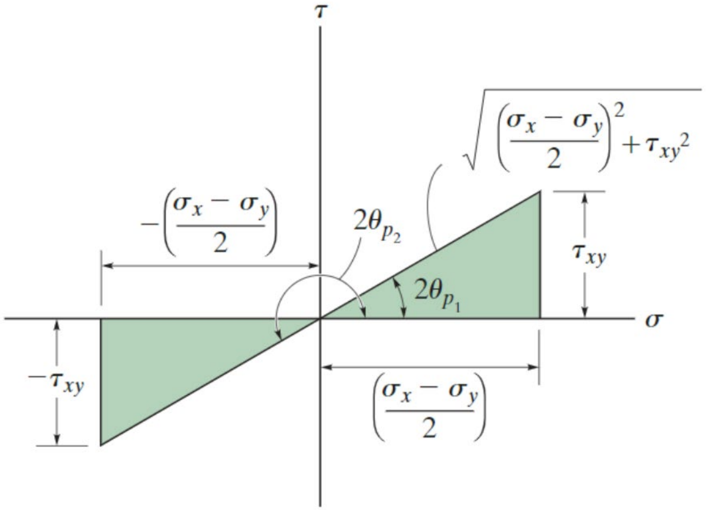


Fig. 9–8

Principal Stresses

What are the maximum/minimum normal stress values (**the principal stresses**) associated with θ_{p1} and θ_{p2}? Start from:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

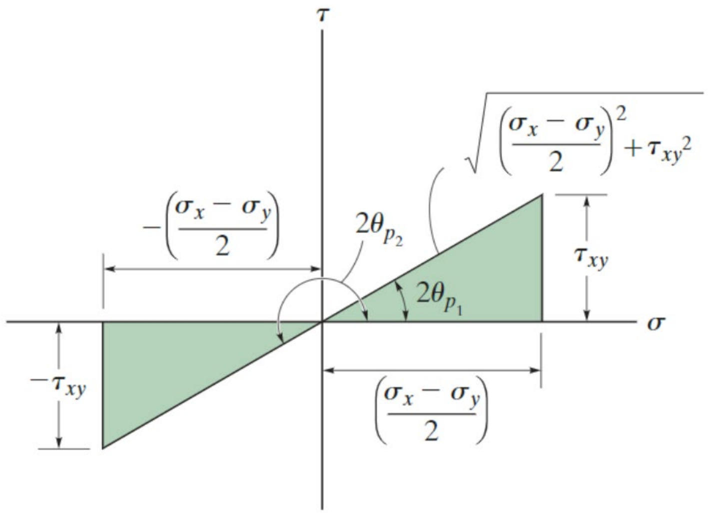
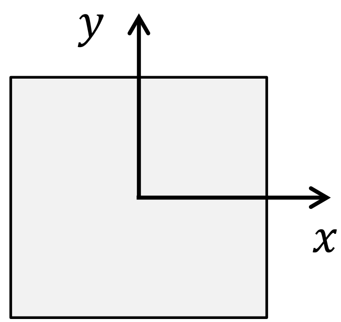


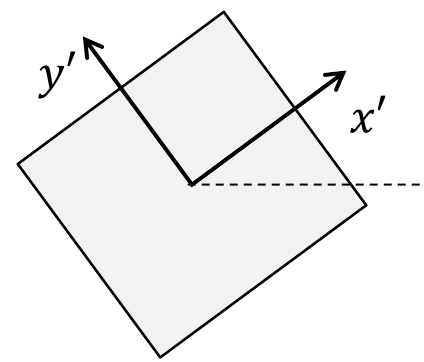
Fig. 9–8

Principal Stress Element

- Rotate original element by $\theta_{p1} \Rightarrow$ maximum stress σ_1 occurs on face originally aligned with x axis
- The angle $\theta_{p2} = \theta_{p1} + 90^o$ defines the orientation of the plane (face) on which the minimum stress σ_2 occurs



Original stress element



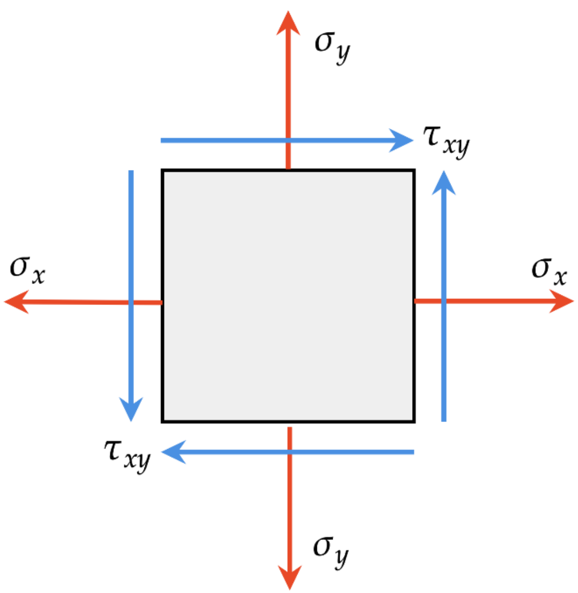
Rotate by θ_{p1}

Principal Stress Element

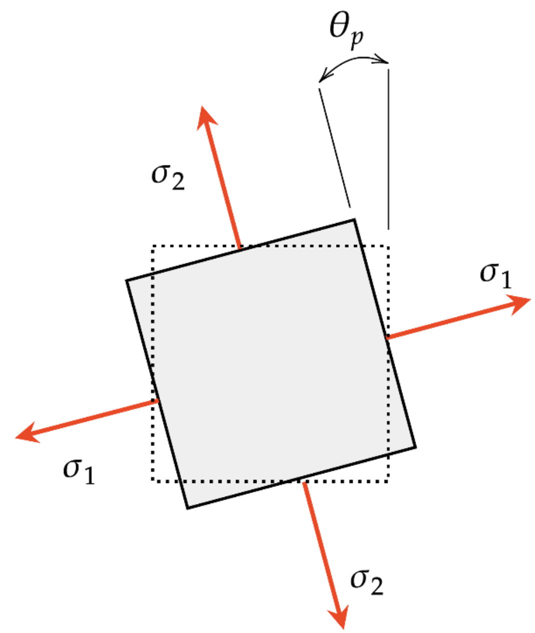
We always use the convention $\sigma_1 > \sigma_2$, i.e. σ_1 is the maximum stress and σ_2 is the minimum stress
→ Note that it is possible that σ_2 is greatest in **absolute value**, i.e. consider $\sigma_1 = -10$ MPa and $\sigma_2 = -20$ MPa

Important: A principal stress element has **no shear stresses** acting on its faces!

→ Try it yourself! Show that $\tau_{x'y'}(\theta_{p1}) = 0$



Original stress element



Principal stress element

Maximum shear stress

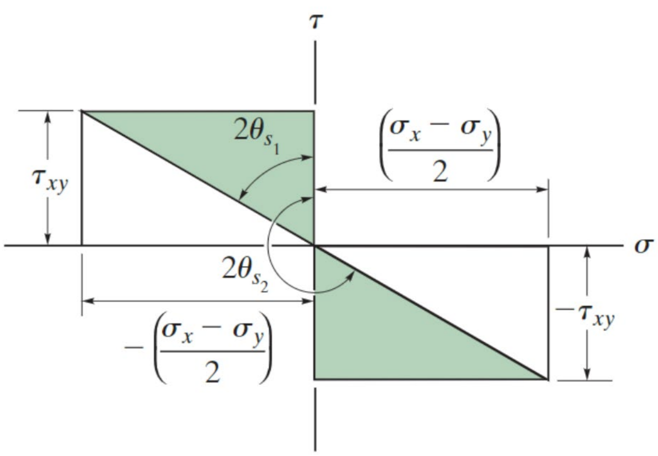
At what angle is the shear stress $\tau_{x'y'}$ maximized? Start from:

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\tan(2\theta_s) = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

There are two roots (that we care about):

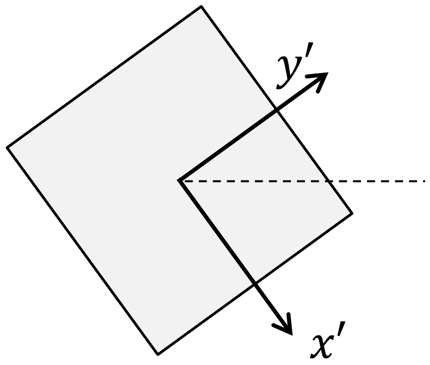
$$\theta_{s1} \text{ and } \theta_{s2} = \theta_{s1} + 90^o$$



Maximum shear stress

What are the **maximum/minimum in-plane shear stress values** associated with θ_{s1} and θ_{s2}? Plug in values of these angles into the expression for $\tau_{x'y'}$ to obtain

$|\tau_{max}| = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ → positive shear stress for θ_{s1}, negative shear stress for θ_{s2}



Rotate by θ_{s1}

Important: a maximum shear stress element has

1) Maximum shear stress equal to value above acting on all 4 faces

1) A normal stress equal to $\frac{1}{2}(\sigma_x + \sigma_y)$ acting on all four of its faces, that is:

$$\sigma_{x'} = \sigma_{y'} = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

3) The orientations for principal stress element and max shear stress element **are 45^o apart, i.e.**

$$\boldsymbol{\theta_s = \theta_p \pm 45^o}$$

Mohr's circle: graphical representation of stress transformations

The equations for stress transformation actually describe a circle if we consider the normal stress $\boldsymbol{\sigma_{x'}}$ to be the x-coordinate and the shear stress $\boldsymbol{\tau_{x'y'}}$ to be the y-coordinate.

All points on the edge of the circle represent a possible state of stress for a particular coordinate system.

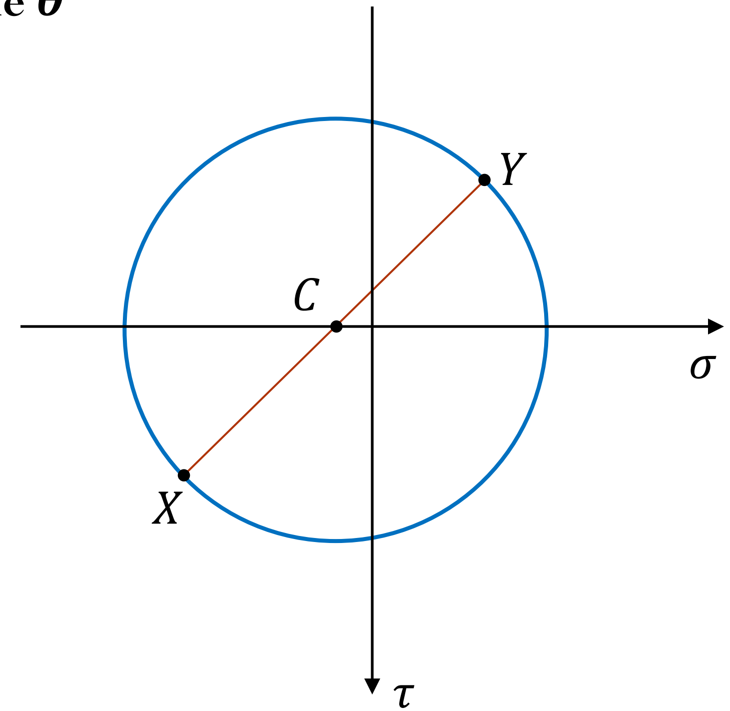
Rotating around the circle to a new set of coordinates an **angle $\boldsymbol{2\theta}$** away from the original (X,Y) coordinate represents a stress transformation by **angle $\boldsymbol{\theta}$**

Circle center location: $C = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_1 + \sigma_2}{2}$

Circle radius: $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Point X: (σ_x, τ_{xy})

Point Y: $(\sigma_y, -\tau_{xy})$



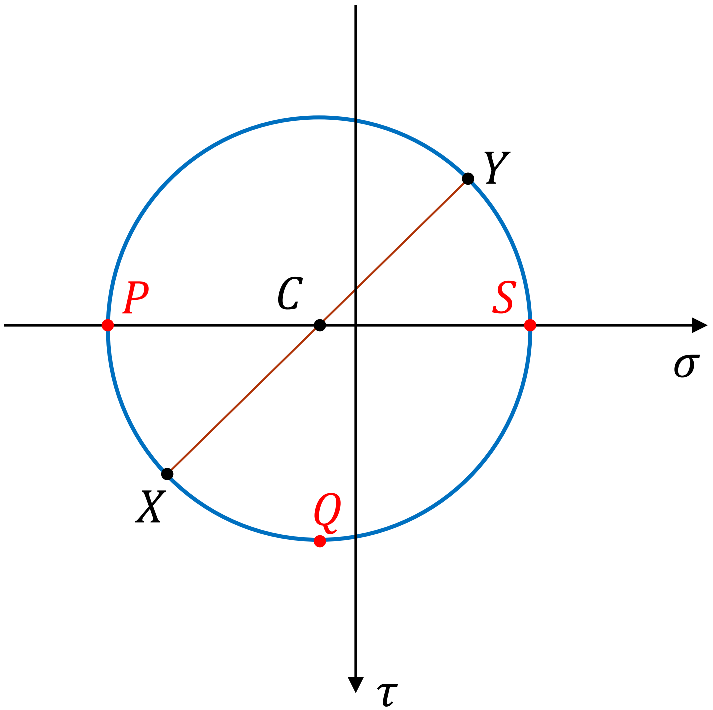
Mohr's circle: graphical representation of stress transformations

Circle center location: $C = \sigma_{avg} = \frac{\sigma_x+\sigma_y}{2} = \frac{\sigma_1+\sigma_2}{2}$

Circle radius: $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Point X: (σ_x, τ_{xy})

Point Y: $(\sigma_y, -\tau_{xy})$



Some questions:

Answer choice	Sign of $\boldsymbol{\sigma_x}$ for this circle	Sign of $\boldsymbol{\sigma_y}$ for this circle	Sign of $\boldsymbol{\tau_{xy}}$ for this circle	Point corresponding to $\boldsymbol{\sigma_1}$	Point corresponding to $\boldsymbol{\sigma_2}$	Point corresponding to $\boldsymbol{\tau_{max}}$
A	Pos.	Pos.	Pos.	P	P	P
B	Neg.	Neg.	Neg.	Q	Q	Q
C	=0	=0	=0	S	S	S

Also: where are angles $\theta_{p1}, \theta_{p2}, \theta_s$ on the circle?

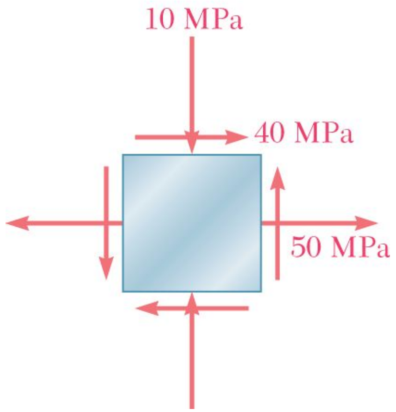
Example: For the state of plane stress shown

a) Calculate the principal stresses and show them on Mohr's circle

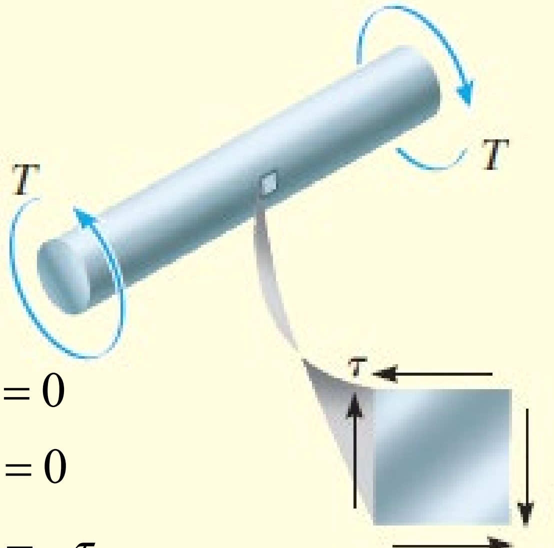
b) Calculate the maximum shear stress and label it on Mohr's circle

c) Calculate the state of stress $\left(\sigma_{x'}, \sigma_{y'}, \tau_{x'y'}\right)$ for a CCW rotation of $\theta = 60^o$; show this state on Mohr's circle

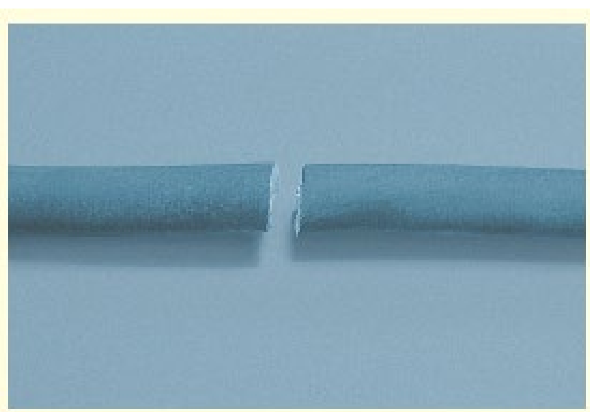
d) Draw the principal and maximum shear stress elements

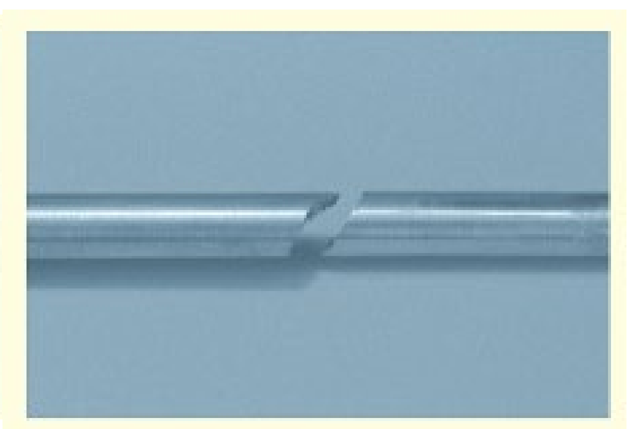


<u>Example:</u> When the torsional loading *T* is applied to the bar, it produces a state of pure shear stress in the material. Determine (a) the maximum in-plane shear stress and the associated average normal stress, and (b) the principal stress.

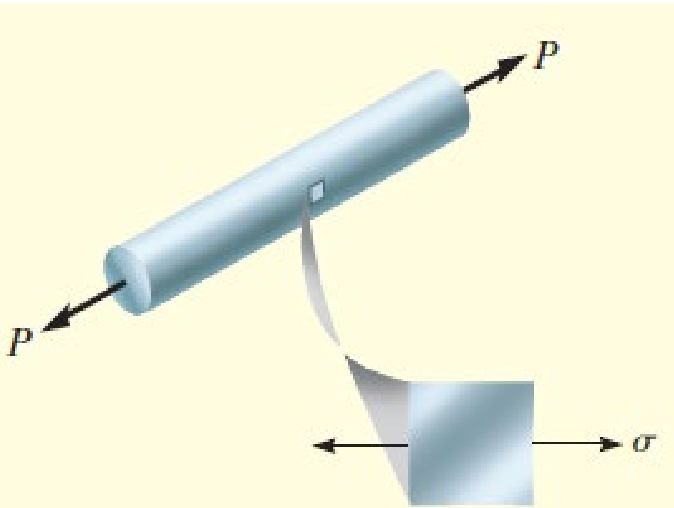


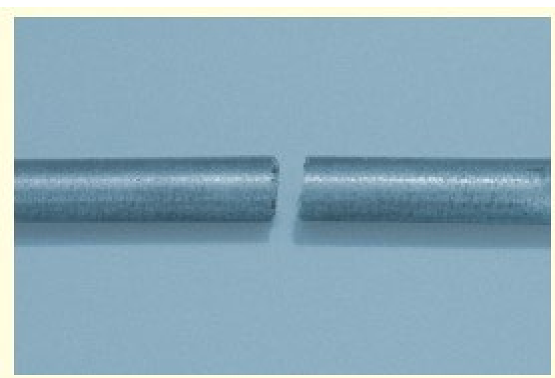
$$\sigma_x = 0$$
$$\sigma_y = 0$$
$$\tau_{xy} = -\tau$$



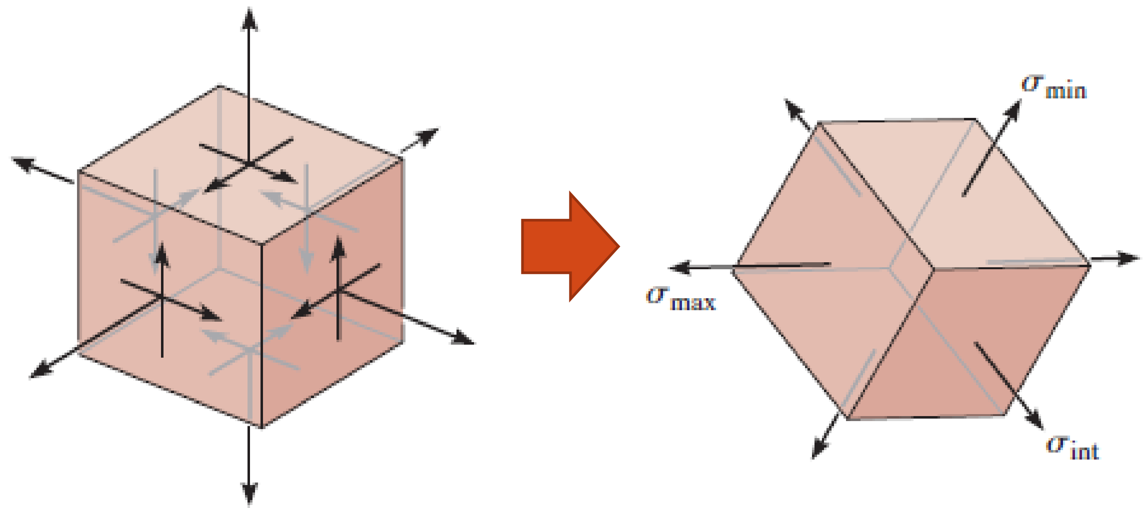


<u>Example:</u> When the axial loading *P* is applied to the bar, it produces a tensile stress in the material. Determine (a) the principal stress and (b) the maximum in-plane shear stress and associated average normal stress.



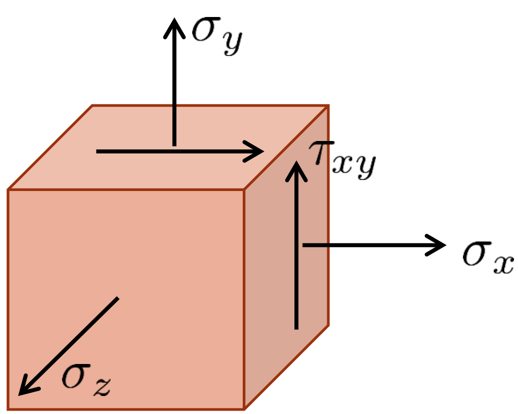


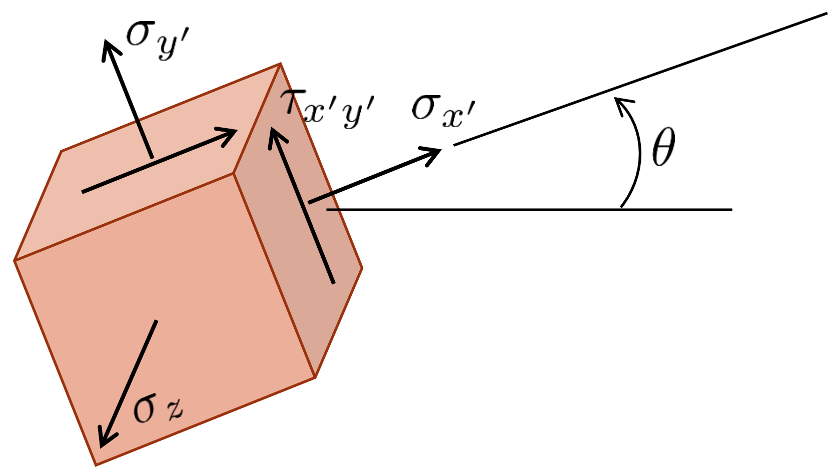
General (tri-axial) state of stress



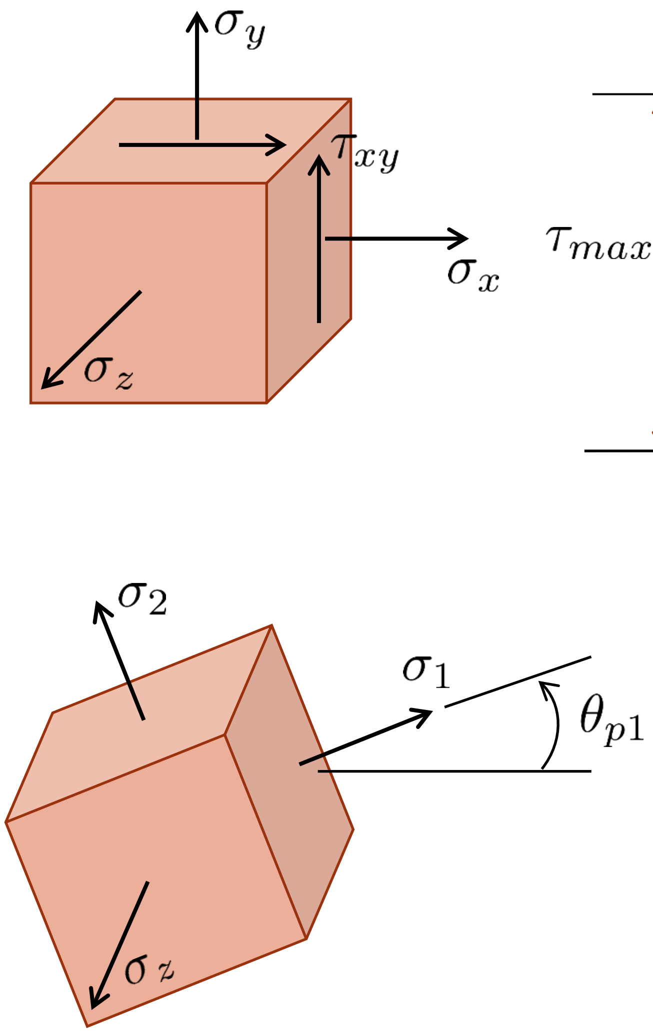
- Three principal stresses
- Corresponding principal planes are mutually perpendicular
- No shear stress in the principal planes

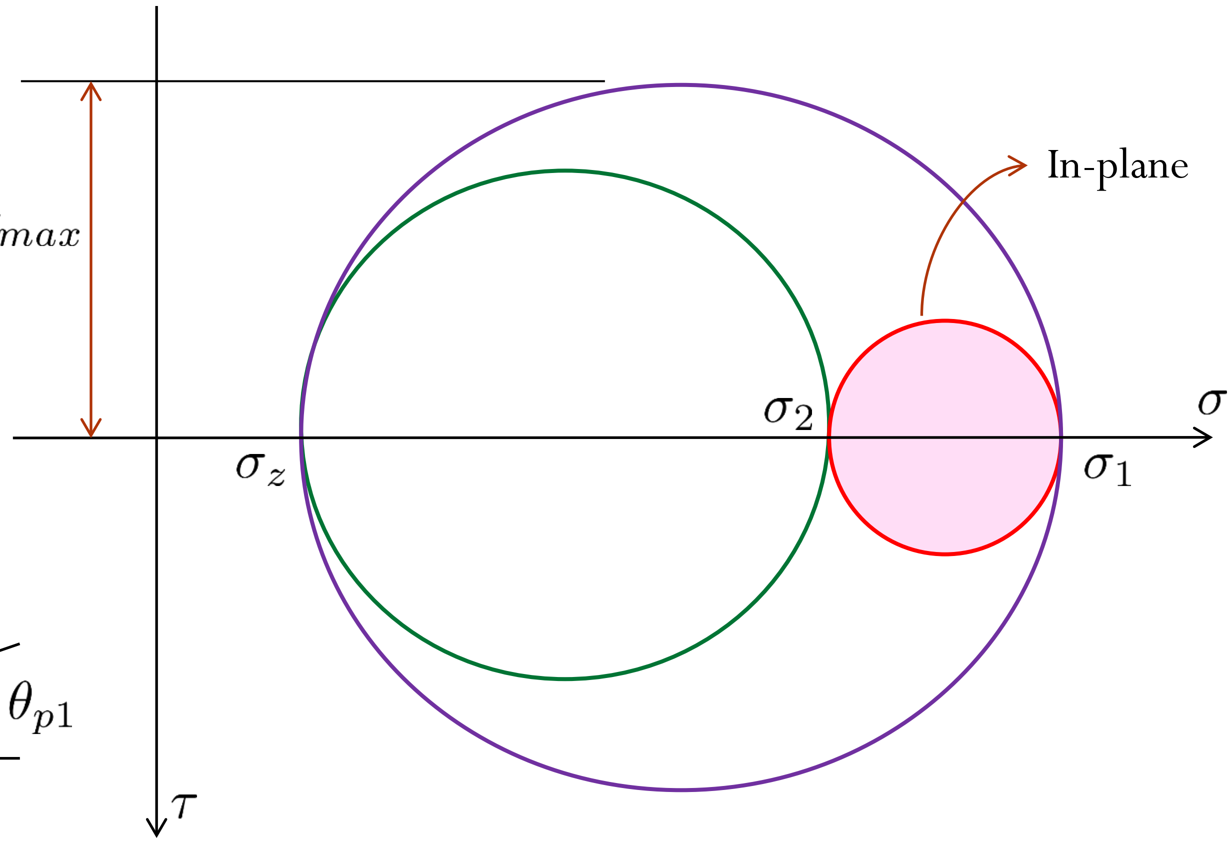
- If we rotate the above element on the right about one principal direction, the corresponding stress transformation can be analyzed as plane stress.





Maximum absolute shear stress





$$\sigma_{\max} = \sigma_1$$

$$\sigma_{\min} = \sigma_z$$

$$\sigma_{\text{int}} = \sigma_2$$

$$\tau_{\max} = \frac{|\sigma_1 - \sigma_z|}{2}$$

Example: For the state of plane stress shown, determine (*a*) the principal planes and the principal stresses, (b) the maximum in-plane shear stress, (c) the absolute maximum shear stress